MEMOIRS OF THE MANCHESTER
LITERARY AND
PHILOSOPHICAL SOCIETY.

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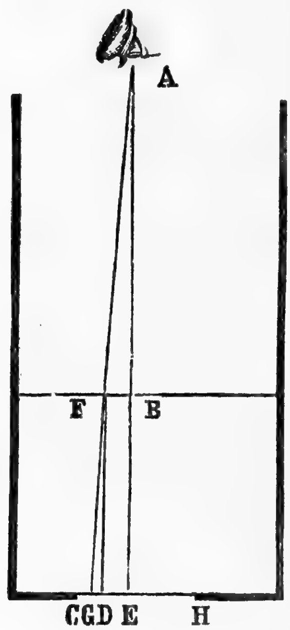
I. *On the Mean Intensity of Light that has passed through Absorbing Media.* By JAMES BOTTOMLEY, D.Sc., F.C.S.

Read October 4th, 1881.

IN colorimetric experiments the areas compared are assumed to be of the same tint throughout. When, however, we look through columns of coloured liquid at external white surfaces, this condition will not be exactly fulfilled.

Suppose the bottom of the containing cylinder to be perfectly flat. Owing to the adhesion of the liquid to the sides, and the consequent elevation there above the general level, the colour will be more intense near the sides. This, however, if the cylinders are of moderate

radius, may be remedied by covering the bottom with a black plate having a small aperture in the centre; in this way the rays coming from the sides are cut off. But, even supposing this done, the colour will not be exactly the same over the whole area. Suppose an eye to be placed at A in the axis of the cylinder, and let C H repre-



sent the section by a vertical plane of the circular area at the bottom admitting light. Let A E be the ray which passes through the centre; the path of this ray through the liquid will be EB. Let AFG be another ray which reaches the eye; the path of this ray through the liquid is F G; and as this is greater than E B, the absorption will be greater. Hence the colour will gradually increase in intensity as we pass from the centre to the circumference. For colorimetric measurements we take the vertical length of the column of liquid. In what follows I propose to consider if this would lead to any noticeable error. Our impression of colour will not be

derived from a consideration of any one point of the area, but from a consideration of the whole area; hence the colour we observe is the mean colour. This, however, although very nearly, will not be exactly the same as at the centre. Suppose G to be the centre of a very small area, which we may denote by ds; then the quantity of light which passes through this small area towards the eye we may denote by ads. Since the area is very small, and ultimately vanishes, we may consider G F as the path of all the rays passing through this small area and reaching the eye. Let G F be denoted by x; then the intensity of the light after passing through the liquid will be $ak^x ds$, k being the coefficient of transmission. For simplicity I shall suppose k the same for every species of light, so that we need only consider one term of the above form. The small element ds we may regard as part of an elementary ring of area $2\pi r dr$, where r denotes G E. The quantity of light passing through this ring towards the eye will be $2a\pi r k^x dr$. If, then, we integrate this between limits 0 and R, and divide by πR^2, we shall obtain the mean intensity. This will be

$$\frac{2a}{R^2}\int_0^R k^x r dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let H be the elevation of the eye above the bottom of the cylinder, and h the height of the column of fluid; also let μ be the index of refraction, θ the angle of incidence, and θ' the angle of refraction. Then we have the relationship

$$\sin\theta = \mu \sin\theta', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$r = h\tan\theta + (H - h)\tan\theta. \quad . \quad . \quad . \quad . \quad (3)$$

We may also write (1) in the form

$$\frac{2a}{R^2}\int_0^R \epsilon^{-mh\sec\theta} r dr.$$

The integration of this expression might be troublesome. Usually θ will be a small angle; suppose, then, that we neglect the cube. From (2) and (3) we deduce with this supposition

$$\theta = \frac{\mu r}{\mu h + \mathrm{H} - h},$$

and the integral may be written in the form

$$\frac{2a}{\mathrm{R}^2}\int_0^{\mathrm{R}} \epsilon^{-mh(1+pr^2)} r\,dr,$$

where p has been written for

$$\frac{\mu^2}{2(\mu h + \mathrm{H} - h)^2};$$

we may also write it in the form

$$\frac{2a}{\mathrm{R}^2}\,\epsilon^{-mh}\int_0^{\mathrm{R}} \epsilon^{-mphr^2} r\,dr.$$

The integral taken between the assigned limits is

$$\frac{a\epsilon^{-mh}}{m\mathrm{R}^2 ph}\left(1 - e^{-mph\mathrm{R}^2}\right).$$

If we expand the term $e^{-mph\mathrm{R}^2}$ and neglect terms containing the fourth and higher powers of R, we shall obtain for the mean intensity

$$ae^{-mh}\left(1 - \frac{mph\mathrm{R}^2}{2}\right).$$

If, as is usual, H be large compared with R, the mean intensity would differ very little from the intensity of the central ray. An examination of the term $\frac{mph\mathrm{R}^2}{2}$ will show if a correction is necessary in any case.

II. *Correction of the Formula used in Photometry for Absorption when the medium is not perfectly transparent.* By JAMES BOTTOMLEY, D.Sc., F.C.S.

Read October 4th, 1881.

SUPPOSE we have a column of length l, containing q units of colouring-matter per unit of length, the medium being perfectly transparent; then the light transmitted will be ak^{ql}. Now suppose the colouring-matter, instead of being uniformly diffused through the whole column, to be confined to an extremely small section at the bottom, of length l', so that the length of the column above the section may be still taken as l. The intensity of the transmitted light will be the same in both cases; hence $k^{ql}=k^{q'l'}$.

If the medium be not transparent, we may now suppose the column of length l above the section containing the colouring-matter to be occupied by some medium that absorbs light. Let ρ be its coefficient of transmission. The light incident on the bottom of the column is of intensity $ak^{q'l'}$. After penetrating the column the intensity will be $ak^{q'l}\rho^l$. But by what precedes, $q'l'=ql$; so that the intensity of the transmitted light corrected for absorption by the medium will be $a(k^q\rho)^l$. If, then, we have two cylinders containing q and q' units of colouring-matter per unit of length, and columns of liquid l and l', we shall have the relationship $(k^q\rho)^l=(k^{q'}\rho)^{l'}$. From this equation we may determine ρ in terms of k and known quantities thus:—

$$\rho=k^{\frac{q'l'-ql}{l-l'}}.$$

In some experiments on the absorption of light by carbon diffusions I noticed that when one diffusion was much

stronger than the other there was a slight departure from the simple rule of colorimetry which held in other cases; this I thought might be due to the absorption of the water employed. The probability that this is the cause would be increased if the value of ρ deduced from one experiment, being applied as a correction to the other experiment, gave consistent results.

If we write ρ in the form k^p, the formula to be used may be written $(k^p k^q)^l = (k^p k^{q'})l'$. This leads to the relation $(q+p)l = (q'+p)l'$.

The standard solution contained 1·2 cub. c. of a strong carbon diffusion in 500 cub. c. of water; and the length of column was 21·2. Comparing with this a solution containing 9·6 cub. c. in 500 cub. c. of water, on one occasion I made 2·94 the equivalent column, on another occasion 2·87, and on a third occasion 2·8. The mean of these three results is 2·87. Hence the corrected formula will be

$$(Q+0·114)l = (Q'+0·114)l',$$

Q and Q′ denoting the number of cubic centimetres of the strong diffusion added to each cylinder. In the following table I have recalculated the results given on page 197, vol. xix. of the 'Proceedings':—

A.	B.	C.
2·02	1·92	1·77
1·59	1·44	1·32
1·22	1·15	1·06
1·05	0·96	0·88
0·89	0·82	0·76
0·78	0·72	0·66

A denotes length of column by experiment, B denotes theoretical length calculated by corrected formula, C the theoretical length deduced from uncorrected formula. It will be noticed that the discrepancies between A and B are less than those between A and C.

III. *On the Failure of certain Mathematical Solutions of the Problem of the Motion of a Solid through a Perfect Fluid.* By R. F. Gwyther, M.A.

Read October 18th, 1881.

Of the solutions with which I intend to deal, we may take that by Stokes for the motion of a sphere as the type. In his paper (Camb. Trans. vol. viii. and reprint) Stokes considers to what degree his solution differs from the results of experience, and discusses the origin of this divergence. I propose to show that the origin is possibly of a different nature to any there discussed. Stokes's solution may be stated thus: If A be the centre of the sphere, of radius a, AX its direction of motion at time t, (r, θ) the zonal coordinates of any point in the fluid referred to AX as axis, and if ϕ be the velocity-potential of the motion,

$$\phi = \text{const} - \tfrac{1}{2}\frac{Va^3}{r^2}\cos\theta,$$

and

$$\frac{p-P}{\rho} = \tfrac{1}{2}\frac{a^3}{r^2}\frac{dV}{dt}\cos\theta + \frac{V^2a^3}{2r^3}\left\{(3\cos^2\theta - 1) - \frac{a^2}{4r^3}(3\cos^2\theta + 1\right\} \quad (1)$$

giving the fluid-pressure.

To deduce from this the impulsive pressures due to an impulsive change in the velocity of the sphere, we write τ for the short time of action of the impulse, multiply equation (1) by dt, and integrate from $t=0$ to $t=\tau$. Let V′

be the final value of V. The last term in p will contribute nothing to the final equation, and we obtain merely

$$\frac{\pi}{\rho}=\int_0^\tau \frac{p}{\rho}\,dt=\tfrac{1}{2}\frac{a^3}{r^2}(\mathrm{V}'-\mathrm{V})\cos\theta. \quad . \quad . \quad . \quad (2)$$

Now, the motion in question having been produced from rest in an infinite liquid by the motion of the sphere, the motion is the same at every instant as that impulsively produced from rest by giving instantaneously to the sphere its velocity V (Thomson and Tait, vol. i. part 1, page 328; Kirchoff, Vorlesungen, chap. xix. &c.). The value of the impulsive pressures due to the instantaneous production of the velocity V of the sphere, and which produces the consequent velocities in the fluid is

$$\frac{\pi}{\rho}=\tfrac{1}{2}\frac{a^3}{r^2}\mathrm{V}\cos\theta,. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $\frac{\pi}{\rho}$, of course, only differs from ϕ by a constant.

Now, there need be no difficulty in supposing a fluid capable of transmitting a tension, provided this tension tends to produce no discontinuity of motion or disruption between near parts, as in the present case. But at the surface of the sphere we get

$$\frac{\pi}{\rho}=\tfrac{1}{2}\,a\,\mathrm{V}\cos\theta;$$

and thereupon a maximum of cohesion is needed of the order $a\rho$V at the extreme rear of the sphere, which is double the mean value required *. Considering the coefficient of cohesion as constant, all other things being alike,

* The pressure in front or tension behind a sphere of radius 1 foot, started impulsively with uni of velocity in water, would be roughly represented by that due to a slab of granite ½ inch thick falling through 3 inches.

the velocity at which Stokes's solution would fail is inversely as the radius of the sphere, if failure can take place through want of cohesion *.

It becomes necessary now to consider the result of such a failure. In a perfect fluid, the consequence must be discontinuity, since we cannot admit any minimum value of ϕ, and therefore of π, within the fluid. In a real fluid, on the other hand, the possibility of the fluid under the tension failing to exert pressure equally in all directions would have to be considered as intermediate between the usual problem and that of disruption. In either case vortex motion would ensue behind the solid.

IV. *On the Manufacture of Salt in Cheshire.*

By Thomas Ward, Esq.

Read November 1st, 1881.

The manufacture of salt has been carried on in Cheshire from the times of the Romans; and as the brine-springs in several places rose to the surface or nearly so in early times, it is quite possible that the Britons may have utilized them. We have no record of salt-making during the early Saxon times, though doubtless it existed; for in Domesday Book

* Or, otherwise, consider the sphere to be brought to rest; the whole motion of the fluid will then cease. But in order to produce rest an impulsive tension of the magnitude named must be sustained at the surface of the sphere.

we find distinct records of salt-works at the three "Wiches," Nantwich, Middlewich, and Northwich, and a reference is made to "King Edward's time" (*i. e.* Edward the Confessor's). In 1132 Hugh Malbanc, in the foundation-deed of Combermere Abbey, says "I grant to the same monks the fourth part of the town of Wych and tithe of my salt and the salt-pits that are mine," &c.

From this period to the commencement of the 16th century the records are very scanty, but sufficient to show that the manufacture was still carried on at the three "Wiches." At this period the most important salt-town was Nantwich or Wich Malbanc. Leland, in his 'Itinerary,' and Camden, in his 'Britannia,' both mention the Cheshire salt-manufacture; and during the 17th century references are frequent, especially in the 'Philosophical Transactions of the Royal Society.' Rock salt was discovered in 1670; and in 1721 the river Weaver was made navigable from Frodsham through Northwich to Winsford. After this period the manufacture steadily but not rapidly increased until 1825, when the duty was taken off salt. Immediately a great advance was made, which continued till 1844, when the East-Indian market was opened to English salt and the manufacture grew still more rapidly. The alkali trade caused another rapid advance, so that at the present time the quantity manufactured is fully ten times as large as at the commencement of the present century.

The districts of Cheshire in which salt is made are in the valleys of the Weaver, Dane, and Wheelock: and this has always been the case; for, with the exception of a small manufacture of salt at Droitwich for a limited time, none has ever been made except in close proximity to these streams. The Weaver valley is the most important; and at Winnington, Anderton, Marston, Wincham, Northwich,

Leftwich, Winsford, and Nantwich salt is now being made or has been made in times past. Since 1847 Nantwich has ceased to manufacture salt.

Middlewich is at the junction of the Croco with the Dane, which latter stream is a tributary of the Weaver. In the neighbourhood of Sandbach, at Wheelock, Lawton, and the surrounding district, salt is made. These places lie in the Wheelock valley, a tributary of the Dane.

The Cheshire salt-beds lie in the Keuper marls, though they are not coextensive with these marls. The red or Triassic marls of Cheshire lie in a kind of basin compared to an elongated saucer with its longest axis lying in a nearly north and south direction. The best-known and most important beds of rock salt are about the centre of this basin, in the neighbourhoods of Northwich and Winsford. At Lawton, in the south-east corner of the basin, beds of rock salt have been found at a considerable height above sea-level. At Northwich and Winsford the rock salt lies below the level of the sea. The Keuper marls of Cheshire are covered by drift. The clays, gravels, and sands of the drift are very much mixed up; and the clay is full of boulders of granite, and various kinds of stone, many of the softer kinds being deeply ice-marked or scratched.

In the early history of the salt-trade, when but a very small quantity of salt was made, the springs at Northwich, Middlewich, and Nantwich either gently ran away into the rivers or rose nearly to the surface. When the rock-salt was discovered near to Northwich in 1670, a strong brine was found running over the surface of the salt. This brine was utilized at once, being stronger than that of the natural springs. On the banks of the Weaver many brine wells were sunk; and since that time all the white salt manufactured has been made from the brine thus discovered. Neglecting minor thin seams of salt which are

met with either above or below the main beds, we may say there are two thick beds of rock salt, known locally as top-rock salt and bottom-rock salt. These two beds are separated by a layer of marl much indurated and containing veins of salt running nearly vertically, as if occupying rifts, or cracks, or crevices in the hardened marl. This layer is about 30 feet thick. The first bed of rock salt, or the "top rock," is at Northwich from 40 to 80 yards from the surface, varying with the different surface-levels and dipping from N.E. to S.W. The surface of this salt bed is very irregular, being waterworn and channelled as if by miniature streams. In most cases, immediately before reaching the salt a much indurated marl is found, locally termed "flag." On piercing this flag brine was met with in the first instance, and continues so to be to the present day *.

Brine is formed by fresh water reaching the surface of the salt either at the so-called outcrop or through fissures and faults in the marl. The marl itself, when unfissured or undisturbed, is perfectly impermeable. It is pretty certain that the water gains admission at a higher level than the rock salt; for the brine originally rose nearly to the surface of the ground in some districts, and in others rose many yards up the shaft when the "flag" was pierced. Brine is only met with on the surface of the top-rock salt, which salt averages about 25 yards in thickness in the neighbourhood of Northwich. At Winsford the top-rock salt lies more than 60 yards from the surface, and is rather thicker than at Northwich, though the indurated clay separating the two beds of salt is of about the same thickness. The bottom bed of rock salt is about 35 yards thick at Northwich, and rather more at Winsford.

* At Nantwich this "flag" has been met with in a boring for brine now being carried on.

No water is ever found on this salt; the "rock head,' as it is called, is perfectly dry.

It is almost impossible to say what the area of these beds of rock salt is. Judging, however, by the subsidence of the land, and taking into consideration the furthest points at which salt has been proved by shafts, I have come to the conclusion that the Northwich beds occupy at least three square miles, and contain in the aggregate about 900,000,000 tons—the upper bed 380,000,000, the lower 520,000,000. The Winsford district is more difficult to determine; but, being again guided by the distinct subsidences of the land, there cannot be less than six miles of salt beds, containing at least 1,900,000,000 tons of salt. It is quite impossible to estimate the quantities of rock salt in the Nantwich, Middlewich, and Lawton districts, as there are no mines, and only in Lawton has the salt rock been bored through. There are several other districts where salt must exist, as is shown by the subsiding of the land. The figures given, which are the merest approximations, show that practically the salt beds are inexhaustible.

The first, or top-rock salt, was discovered in 1670; and very shortly after this mines were worked, but not to any large extent. It was not until 1721, when the river Weaver was made navigable, that the rock salt could be largely utilized. In the year 1732 there were sent down the Weaver 9322 tons of rock salt. The quantity shipped down the river increased year by year until in 1778 it reached 54,000 tons. In 1781 the bottom or lower rock salt was discovered; and all new mines were sunk to it, the quality being better. After the commencement of the present century the top mines ceased to be worked; and now only one is known to exist. Nearly the whole of these top mines were destroyed by the breaking-in of water or brine at the shafts. One

after another they collapsed, leaving large funnel-shaped pits filled with water, locally known as "rock pit-holes," to mark their position. In all the upper mines large pillars, about 5 yards square, were left to support the roof. In the lower mines these pillars, though originally of the same size, are now left larger, varying from 8 to 12 yards square. One or two lower mines have collapsed by the roof falling in for want of sufficient supports. Some have been abandoned because of the breaking-in of brine or water. Of late years, however, when worked to the boundary of the owner, they have been allowed to fill with brine, which is pumped out for the manufacture of white salt. About 150,000 tons per year of rock salt are mined from the lower portion of the salt bed, which is the purest. This is not quite one tenth of the quantity of white salt made from brine. Whether, however, the salt is mined as rock salt or is manufactured from brine, the beds of rock salt before described furnish the whole.

The manufacture of salt from natural brine is carried on more largely in Cheshire than in any other portion of the world. The brine is found upon or near to the surface of the first bed of rock salt. It is reached by the sinking of a shaft or huge well, sometimes as much as 10 feet in diameter, which is cylindered to keep out the fresh water. Till recently there has been no great diminution in the quantity of brine. As soon as the "flag" overlying the "rock head" was pierced the brine rose very copiously, and the most rapid pumping only produced a slight change of level. The height of the brine above the surface of the rock salt varied in all the districts, but in every case was considerable, being frequently 50 yards. The level of the brine when at rest or unpumped has fallen very considerably of late years, owing to the enormous increase in the make of salt. At Winsford the level in 1880 was 20 yards

lower than in 1865. It is evident that more brine is being pumped out than fresh water getting in to make new brine to replace it. Many shafts have been completely exhausted.

Of late years a number of old worked-out mines in the lower bed of rock salt have been used as reservoirs, into which the brine from the rock-head runs night and day. These enormous reservoirs are nearly always full; in the majority of cases the brine rises high up the shafts. The mines in the bottom rock vary from 100 to 112 yards from the surface, and the salt is worked out to a depth of about 15 feet. Some of these reservoirs will hold more than 50,000,000 gallons of brine. Every day the trade consumes more than 4,000,000 gallons in making white salt. This enormous consumption has produced very startling results, which I shall refer to later on. The brine is pumped up by means of powerful steam-engines into reservoirs or tanks, and from these it is distributed to the pans through pipes.

The specific gravity of fully saturated brine is about 1·2; and it contains 26 per cent. of salt. Roughly speaking, the Cheshire brines consist of 1 part salt, 3 parts water. The business of the salt-manufacturer is to drive away the water and to retain the salt. This is done by heat; and the *rationale* of the process of making salt is extremely simple. When brine is fully saturated there is a state of equilibrium, as it were. If we reduce the quantity of water, this balance is disturbed, and a portion of the salt crystallizes out, until the brine is again at saturation-point. If we continue driving off the water, the salt continues to crystallize and deposit. This process might go on till all the water was expelled and all the salt crystallized; but in the process of manufacture this never occurs, as the pan is replenished with brine from time to time.

During the manufacture of salt the laws regulating the formation of salt crystals may be clearly seen. The kinds

of salt manufactured do not vary at all in their constituent parts, but merely in the size of the crystal or "grain" as it is called. To sum up the whole process in a few words, we may say generally that the larger the crystal the less the heat and the longer the time required to make the salt; the smaller the crystal, the greater the heat, and the less time required to make the salt. Brine boils at 226° Fahrenheit. Boiled salts are taken out of the pan two or three times in twenty four hours; common salt, such as is used for soaperies, chemical works, &c., every two days. Fishery salt remains in the pan, according to the grain, from six to fourteen days; bay salt three weeks to a month. The manufacturer, by manipulating his brine, can make the crystal more or less flaky or more or less solid as he wishes; but the general principle is not altered. Foreign ingredients, such as gelatinous matter or alum, have very peculiar effects on the brine and on the salt crystal. Bay salt is made at a temperature of about 90°, fishery from 90° to 140° according to grain, common salt 170° to 180°. There is no exactitude maintained on this point; nor is it material, as a slight variation in the grain makes no difference in the sale and use of the salt.

The manufacture of salt is extremely simple, almost rude; yet the simple plan in use for ages has been found to answer best in the long run. The price of salt is so low, and the manufacture such a bulky one, that costly and elaborate apparatus has never been found to answer. Comparing the life of an ordinary open salt pan, such as is used in the trade almost universally, with that of any of the numberless patented pans that have been tried, it has been found that it costs less to manufacture salt by it, and that the pan is far easier to repair.

The pans used in the manufacture of salt are made of wrought iron, and riveted together like boilers. Under

each pan are three or four fires, extending about five feet from the front of the pan. The pans are set upon brick-work; and the flues are continued under the whole length of the pan, and thence lead to the chimney. Occasionally there is a chimney to each pan; but more often one chimney answers for several pans, as many as sixteen fires often being connected with one chimney. The pans used for boiling the fine salts rarely exceed 33 to 35 feet in length, or the brine could not be kept boiling all over the pan. The surplus heat, which in the coarse-salt pans would be utilized to make salt for another 30 or 40 feet, is passed under what is called the hothouse, where the boiled salt which is made into lumps or loaves is dried. Over this stove is a loft or storing-room, where the lump salt is kept dry and warm. The quantity of fuel consumed in making salt is very large. For making a ton of common coarse salt about 10 cwt. of slack or other cheap fuel is used. For making a ton of boiled salt about 13 cwt. of "burgey" or "through" coal is used. It is found that ordinary slack will not get up and maintain heat enough to keep the pans boiling.

The number of pans now in existence in the various salt-districts is as follows:—

Winsford	638	
Northwich	458	(At Droitwich 151 pans.)
Middlewich	13	
Sandbach	69	
	1178	

The quantity of white salt manufactured annually in each district is, in round numbers:—

Winsford	1,000,000	tons.
Northwich	600,000	„
Middlewich	20,000	„
Sandbach	100,000	„
	1,720,000	„

The first pans used in the manufacture of salt of which we have any record were made of lead. A sheet of lead about $\frac{1}{4}$ inch thick and 2 feet 8 inches square, in the case of a pan in my possession, was obtained, and bent up at the sides, and the corners hammered together. This pan was from 3 to 4 inches deep *. Six of these pans were usually set over flues in a small room. The contents of these six pans, or "leads" as they were called, formed the unit of measurement in the salt-trade when every maker was regulated by laws as to quantity of brine, time of taking it, hours of drawing salt, &c. So strict were the laws that an officer (called a "pan-cutter") was employed to see that all the pans conformed to the standard pan; if any were larger, he was to cut them down. After a time four pans took the place of the six, and were of equal capacity in the whole. Dr. Jackson, about 1668, describes these four pans as made of iron and superseding the six leaden ones. Their size, he says, was about a yard square and about 6 inches deep, and they held 28 gallons. In Dr. Brownrigg's time, 1765, the pans held 800 gallons; and Jars says the largest pans at Northwich in 1765 were 20 feet long by 9 or 10 wide, holding about 1100 gallons. Now the small pans used for boiled salt, which are the smallest in the trade, are from 25 to 35 feet long by from 20 to 24 wide, and 15 to 18 inches deep, averaging 5000 gallons to the pan. The pans used for making coarse-grained salt vary from 50 to 70 feet in length (some few being even longer than this) and are about 25 feet in width, the depth being 18 inches. These would contain over 16,000 gallons. The growth in the size of the pan has been equalled by the growth in the

* There is an old lead in Warrington Museum, found at Northwich. It measures 3 feet 8 inches in greatest length, and 2 feet 8 inches in width, being 4 inches deep. It would hold about 20 gallons at most. The old pan in my possession, which is 25 inches square by 3 inches deep, would only contain about 7 gallons.

manufacture of salt; and most probably the increasing demand stimulated the maker to increase the size of the pan.

We have no statistics showing the quantity of salt made in early times. Judging, however, from the size of the pans or leads, there could not be much manufactured. The possession of a house containing six leads was of quite sufficient importance to be mentioned in Domesday Book as belonging to certain towns or villages. Many of the salt-houses in the "Wyches" belonged to noblemen, and appear to have been used to make salt for their houses; for we read, "There is in Wych half a salt-house to supply the Hall." In 1605 we read in an old letter, "There is in the said Towne (Northwich) one hundred and thirteen salt houses every one containing four leads apeece . . . and one Four leads which was given to the Earl of Derby . . . for the portion of his house."

From an estimate made about 1675, by William Lord Brereton, it would seem that about 305 tons 7 cwt. of salt were made at Northwich weekly; 107 tons 10 cwt. at Middlewich; and 105 tons at Nantwich. Thus the total Cheshire make would be about 26,927 tons per annum. The whole of this was for the home trade. In 1732 only 5202 tons of manufactured salt were shipped down the river Weaver, made navigable in 1721. None of this came from Middlewich and Nantwich, but from Winsford and Northwich. In 1801 this had reached 142,675 tons per annum. In 1820 we find 186,666 tons sent down the Weaver. In 1825 the salt-duty was taken off; and in 1830 the quantity sent down the Weaver reached 312,012 tons in the year, and in 1840 414,156 tons. In 1844 English salt was allowed to be sent to the East Indies; and the result is seen by the exports down the Weaver reaching 607,395 tons in 1850. During the next ten years the

increase was not large; but as soon as the alkali-trade began to extend, the manufacture of salt increased, and in 1870 we find 991,158 tons sent down the Weaver, which reached 1,087,214 in 1880. It will be seen from these figures how rapidly the salt-trade has increased in the last half century. As the Weaver is only one line of communication (though the chief one) by which Cheshire salt leaves the country, there being canals and railways taking large quantities, the figures above given do not show all the trade. The following table of salt exported from the Mersey ports (viz. Liverpool, Runcorn, and Weston Point), which I have compiled from the official statistics supplied to me for the ten years ending December 31st, 1880, will show the countries taking Cheshire salt, and the quantities supplied to each:—

Countries.	White salt. tons.	Rock salt. tons.
United States (North and South)	2,118,656½	18,827
British North America	691,186¼	31,354¾
West Indies and Central America	40,003¼	757½
South America	34,640	2,258¾
Africa	246,429	746¾
East Indies	2,552,856¼	155
Australia and New Zealand	139,510¼	26,943
Germany	345,229	1,026
Russia	581,501¾	23,697
Norway, Sweden, Denmark, and Iceland	197,299½	39,636
Belgium and Holland	81,633	644,471
France and South of Europe	17,699	943½
England	840,285	135,371
Ireland	469,310	64,155
Scotland	711,229½	19,631
Total	9,067,468	1,009,973¼

This table is a very interesting one; but I must not discuss it, as it affords material enough for a paper in itself. Besides the salt thus exported, we must reckon at least

500,000 tons of white salt and 30,000 tons of rock salt sent into the interior by canal and railway yearly. During the last three years the make of white salt in Cheshire cannot have been much short of 1¾ million of tons per annum.

It may be interesting here to speak of the price at which salt sells at the works. In using the expression "at the works" it must not be understood that salt is sold delivered at the works. The usual rule is to sell delivered at the port of shipment, or, if in the country, at the place where required; but to avoid all complications arising out of varying rates of carriage and freight, it is customary to sell at a "works" price and add on the freight or carriage. The salt that generally regulates prices is common salt, which is the cheapest and most largely used of any. Taking the price of common salt as a standard, the other salts, as a rule, vary according to the following table. When prices rule high, the difference increases; and when they rule low, it is rather less; but on the average the prices here given may be considered the usual ones.

With common as the standard,

		s.	d.	
Butter (or unstoved boiled) salt is		1	6	per ton more.
Shoots, or broken stoved boiled salt		3	0	" "
Handed squares stoved salt	5*s*. to	6	0	" "
Fishery or coarse salt (according to quality)	1*s*. 6*d*. to	5	0	" "

Bay salt and refined table-salt, being made only in very small quantities, fetch prices not regulated by any other qualities, varying from 30*s*. to 50*s*. per ton.

The following table will show the fluctuations in common salt (or the standard salt) for ten years :—

	Prices per ton.					
	Average.		Highest.		Lowest.	
	s.	d.	s.	d.	s.	d.
1871	6	1	7	0	6	0
1872	12	4	20	0	7	0
1873	14	8	15	0	12	0
1874	10	0	12	0	8	0
1875	8	6	9	0	6	6
1876	6	5	8	0	5	0
1877	5	6	7	0	4	6
1878	6	6	7	0	5	0
1879	5	6	7	0	4	6
1880	5	6	6	6	4	6

During 1881 the price has been, on the average, only 4*s*. 9*d*., and stoved salts for India have ruled lower than at any former period.

It would be interesting to point out the various kinds of salt made, the purposes for which made, and the countries to which sent—also to show the bearing of the salt-trade upon the general trade of the country, and more especially upon the shipping trade of Liverpool. This, however, must be left for another paper, if thought desirable.

The most remarkable feature in connexion with the manufacture of salt is the extensive subsidence of land, and the great destruction of property caused by it. In mining the rock salt in the usual way (that is, by blasting and picking) an ordinary mine is formed, subject to the usual mining-accidents of water breaking in or the roof falling. There is no bad gas to cause explosions. It is not accidents of this kind that are referred to in speaking of land-subsidence in the salt districts. The whole of the manufactured salt is literally mined by water. A constant stream of water running over the rock salt to the pumping-stations becomes saturated with the salt that it takes up on its way. Being pumped up, the salt is gained by the method of evaporation before described. It is evident that in gaining the salt in

this manner, by water, no provision can be made for the support of the roof of overlying earths, as is done by pillars in the ordinary mine. The brine streams, or brine "runs," as they are called, resemble brooks and rivulets into which the waters run from all sides and drain the surface. The course of these underground streams is marked by a corresponding depression on the surface of the ground. The land sinks more rapidly directly over the brine-runs than elsewhere, though there is a gradual lowering of the surface of the rock salt generally, even out of the line of the runs—exactly as there is a gradual lowering of the surface of the land by the solid earths and sands carried away by the water draining off it. The salt, however, being so much more soluble than earths, the process is much more rapid. In some cases the overlying earths follow the gradually wasting rock salt continuously, and the subsidence on the surface is regular and continuous; in other cases, where there happens to be a tenacious "flag" or other stiff and solid earths, the superincumbent mass remains suspended over the brine-run for a long time. As soon, however, as the weight of the suspended mass overcomes the tenacity of the marl or flag, a sudden sinking of the whole overlying earths takes place, and a large hole reaching up to the surface is formed, which in time becomes filled with water.

In recent years a number of these holes have been formed in the neighbourhood of Winsford and Northwich. In the neighbourhood of Northwich they have been formed chiefly over old mines. Instead of the hollow being made naturally, as near Winsford, by the ordinary brine-run, it has been made artificially by the miner. In both cases, however, the sudden sinkings are directly caused by the pumping of the brine. Whenever any sinking, whether gradual or sudden, occurs in the neighbourhood of a brook or river (and the majority occur in such positions), the water soon

fills the hollow thus formed, and tiny lakes commence, which go on increasing. The local name for these lakes is "flashes." At Northwich the "top of the brook," as it is called, now covers about 100 acres. At Winsford the top and bottom flashes cover about the same area, whilst both at Winsford and Northwich new lakes are rapidly forming. Near Winsford one called the Ocean covers about eight acres. At Marston, near Northwich, there is one covering ten acres when the water is at its greatest height. On Dunkirk, near to Northwich, the scene of the great subsidence of December 6th, 1880, the hole then formed is rapidly extending, following the course of the Peover brook. About two miles from Northwich, at Billinge Green, away from any salt-works, subsidence has begun and is rapidly extending, a lake having commenced which is only about an acre in extent, though more than 100 acres show signs of subsidence. In the Winsford and Northwich districts more than 1000 acres in each show distinct traces of subsidence; and in both cases the townsand buildings suffer to a most serious extent. How seriously owners of property suffer will be seen from the statistics given here, which I have taken from the evidence given before a Committee of the House of Commons in May last:—

Approximate number of persons as owners injuriously affected ...	418
Public buildings damaged	11
Salt-works damaged	22
Slaughter-houses, stables, and out-buildings damaged	40
Warehouses, workshops, &c. damaged	46
Public houses damaged	56
Shops damaged	202
Houses and cottages damaged	931
Total buildings affected	1308
Area of land affected (acres)	2808

The value of the property affected (exclusive of railways

and canals, which suffer seriously) in its damaged state is estimated at £480,000. The amount of depreciation in value is estimated at £168,500. To keep this property in a good state of repair would require on the average £18,000 per annum. In this estimate, besides omitting railways and canals, no account has been taken of the expenditure of the local boards and highway boards in maintaining their roads and sewers—nor yet of the county authorities in the repair of bridges, nor of the gas- and water-companies in repairs of pipes and loss of gas and water. For all this destruction of property there is no legal remedy, and no compensation whatever is paid.

The owners whose property suffered so severely applied to Parliament to obtain a Bill called "The Cheshire Salt Districts Compensation Bill." The object of this Bill was to obtain compensation from the salt-manufacturers for the damage caused by the abstraction of the salt in the form of brine. There was no attempt made to obtain payment for the salt taken, but merely for the damage done to the property overlying the salt. The Bill was not obtained, owing to many serious difficulties that would arise in carrying it out, but chiefly owing to a very ingenious line of defence set up by Mr. De Rance, a geologist connected with the Ordnance Survey Department. He contended that the enormous subsidences at Winsford were produced naturally by the same causes that produced the Cheshire meres, and that the Northwich subsidences were caused by bad mining. In my opinion—based upon a far more extensive examination of these sinkings than that made by Mr. De Rance, and formed on the spot with the sinkings proceeding daily before me for a number of years—Mr. De Rance's theories are wholly untenable. To attribute to ordinary geological causes, which in almost every case work slowly and continuously, the enormous subsidence of the last fifty years,

and more especially those of the last ten years, and which have gone on increasing in the direct ratio of the increase in the manufacture of salt, when there is an artificial cause at work patent to every one, tends to injure science in the eyes of those who know the whole history of these sinkings. The manufacture of salt in Cheshire is interesting archæologically, historically, commercially, and geologically; and it is impossible to do full justice to it in one short paper.

V. *Remarks on the Terms used to denote Colour, and on the Colours of Faded Leaves.* By Edward Schunck, Ph.D., F.R.S.

Read December 13th, 1881.

At the recent Meeting of the British Association held at York, a paper was read by Dr. Montagu Lubbock before the Section for Anatomy and Physiology, on "the Development of the Colour-Sense," which I had the pleasure to hear. The purpose of the author was to controvert the opinion of those who hold that the colour-sense in man was not always what it is now, but that it has gradually been developed, the last stages of this development having taken place within historical times. It is supposed that the human eye was originally only capable of distinguishing black and white, and that the capacity of seeing the various colours of the spectrum arose by degrees, red being the first and blue the last colour to be discriminated. Mr. Gladstone, in a paper published not long ago, goes so far as to say that the ancient Greeks, having no word for blue,

were blind to that colour, and that it is only since their day that human vision has been so far developed as to perceive the more refrangible end of the spectrum. The author of the paper referred to arrived at the conclusion that there is no sufficient evidence to show that the faculty of perceiving colour has been acquired by man within historical times, a conclusion in which Sir John Lubbock, who took part in the discussion on the paper, entirely concurred. Whatever may have taken place in prehistoric times, there can be little doubt I imagine, looking at the remains adorned with various colours in Egypt and elsewhere, that the more civilized nations of antiquity, though they had fewer pigments at their disposal than we have, were quite as capable of distinguishing colours as we are. It may indeed be asserted that, so far as appropriate arrangement of tints in dress and other articles of daily use is concerned, no advance has been made since the time of the ancients, but rather that we have in this respect retrograded. Evolutionists tell us that we may obtain a good idea of what our prehistoric ancestors were, by observing the present state of savage and uncivilized races, a state from which we have, in the course of ages, emerged. So far, however, as the appreciation of colour is concerned no superiority on our part can be discovered; for whoever will, with an unprejudiced eye, compare the harmonious combinations seen in the articles produced by the less civilized nations of India and China, or by the natives of America and Polynesia, with the hideous contrasts and tasteless arrangements so often displayed in our articles of dress and furniture, will probably incline to the opinion that in this respect we may rather be called savages, and that the incapacity for appreciating colour-harmony inherent in the Teutonic branch of the Aryan race has not been removed, but rather intensified by civilization. Much has,

indeed, been done of late to promote good taste in this as in other departments of art, though our progress has probably been retarded by the introduction of various artificial colouring matters, the extreme brilliancy of which acts as a lure to the uncultivated eye.

Though much of the uncertainty which exists as to the precise meaning of the words used by the ancients to denote colour is of a philological kind, part of it is due, I think, to causes which are still in operation.

1. The ancients, having no fixed scale of colour to refer to, such as we possess in the spectrum, were unable to compare any given tint with that which it exhibits in its highest state of purity, whilst we, by means of the fixed standard at our command, and with the assistance of the so-called chromatic circles and other appliances, can determine not only the exact position and shade of any given colour, but also the extent to which it is degraded or rendered impure; and though the general public very slowly adopts scientific terms and methods, still on the whole the tendency in our days is towards exactitude, and vague terms for objects and sensations are more and more falling into disuse.

2. In one respect the ancients must have laboured under the same disadvantage, in determining the value of colour, as we moderns do. We very seldom see one colour alone, but generally two or more in juxtaposition, and contrasted; and by contrast the effect of each colour on the human eye is considerably modified. Complementary colours, when seen in close proximity, heighten one another. Green next to red will appear much brighter than when placed close to blue. A colour of average purity will appear dull when compared with a brighter colour of the same hue, while it will seem bright when seen alongside a more dingy shade, and so on. Unless great care be taken, therefore,

we are liable to become inaccurate when describing a colour, though on the other hand it may be doubted whether, if the human eye were so constructed as to see only one of the colours of the spectrum, we should, from the absence of contrast, be able to appreciate that colour correctly.

To the fact that we almost always see colours in contrast must be ascribed the habit which men have of speaking of "beautiful colours." No one who has thought on the subject need be told that a simple sensation cannot be strictly speaking beautiful. It is only by combination, contrast, and harmony of sensations that we arrive at beauty. To talk of a beautiful sound, such as a single note of a musical instrument, would be absurd; it is only a combination of sounds that can be called beautiful. The terms "beautiful smell," "beautiful taste," would cause the most ignorant to smile, though it may be contended that a dish uniting various flavours, or a perfume composed of well-assorted scents might be called beautiful. We look at the spectrum thrown on a screen and say it is beautiful; but it is the effect of the various colours seen in juxtaposition, and the exquisite shading and melting of one into the other, that we admire. When we speak of the beautiful colours of sunset, we forget that a fine sunset is in fact a grand chromatic display. We see the fiery red of the fleecy clouds and the deep blue of the sky contrasted with the green of the foliage and the brown of the tree-stems, followed by a flood of yellow light on a cool grey ground in the heavens, while the gloom of night is settling over the earth beneath; and the eye is pleased and satisfied. Were we to see the sun like a ball of red-hot iron set through a coppery sky over a sea of blood washing a coastline of red rocks overgrown with red seaweed, it is certain we should not speak of the beautiful colours of

sunset. Let any one, to test what I maintain, look at any colour, however brilliant and pure, through a tube blackened inside, and say whether it appears beautiful. It is the great activity of the eye, which during our waking hours is constantly roaming from object to object, seldom seeing the same thing or the same colour for more than a few consecutive moments, that deceives us.

3. Much of the confusion as regards the names of colours arises, as it has doubtless at all times arisen, from the habit, difficult to explain, of using inexact designations, and even applying names to colours which we know to be incorrect. Poets, for instance, call gold red, though it is always yellow. We speak of white wines and red wines, though in reality, as we are well aware, they are yellow and purple; so that a thousand years hence it may be possible for a literary man to say that we were colour-blind, our eyes having not yet acquired the capacity to see yellow and blue, yellow appearing to us colourless and purple red; and he may in support of this assertion quote the line of a distinguished poet now living who speaks of the "costly *scarlet* wine," a term which is still more precise and emphatic than simple red. In the course of an investigation, undertaken a short time ago with another chemist, I found that my collaborateur and myself never exactly agreed as to the names to be given to the colours we saw. The series which he named blue, violet, purple, crimson, red, orange, I called violet, purple, crimson, red, orange, yellow; *i. e.* what was to his eye blue was to mine violet; his violet was my purple, and so on. There was no reason to suppose that our perception of colour differed, the difference was, in my opinion, simply one of terms.

The writings of ancient authors abound with instances of the use of colour-names which are seemingly incorrect. We find in Horace (Book IV. Ode 1) the lines :—

Tempestivius in domo
  Paulli, *purpureis* ales oloribus,
Comissabere Maximi,
  Si torrere jecur quæris idoneum.

Another author says :—

Purpurea sub nive terra latet.
Brachia *purpurea* candidiora nive.

We are told by scholars that in these cases *purpureus* means bright, shining; but it still remains to be explained why a word which generally denotes a positive colour, whatever that colour may have been, comes in a few instances to be applied to white objects such as swans and snow. Of course the definition of a word may be so extended as to include any number of widely different meanings, some of these meanings being perhaps due to its mistaken use by authors. It would probably not be difficult to find passages in modern authors in which the word blue sometimes means green, sometimes violet. I met with a case in point recently on reading again Goethe's delightful autobiography 'Wahrheit und Dichtung.' The author, when a young man, was skating on the river on a bitterly cold winter's day. "I had been on the ice," says he, "since early morning, and was therefore, when my mother later in the day drove up to admire the scene, being only lightly clad, almost frozen. She sat in her carriage wrapped in a *red* velvet fur mantle, which, held together in front with thick gold lace and tassels, looked magnificent. 'Give me, dear mother, your fur cloak,' said I without much consideration, 'I am fearfully cold.' She, too, did not consider long, and the next moment I had put on the cloak, which reaching nearly to my feet, being of a *purple* colour, trimmed with sable and ornamented with gold, suited very well the brown fur cap which I wore." Here it is evident that Goethe calls the

same object first red, then purple, and yet Goethe was not colour-blind; he wrote, as every one knows, a work on the theory of colours.

4. Part of the vagueness and uncertainty attending the terminology of colour may be ascribed to a tendency we are all more or less liable to—that of describing colours in figurative and metaphoric terms. The habit, no doubt, arises from the pleasure we feel in comparing two objects, both of which are agreeable to the sense of sight. We speak of a girl having sky-blue eyes and cherry-red lips, whereas slate-coloured and brick-red would be more correct. How often we hear the expression, "he turned as white as a sheet," whereas the human skin is never under any circumstances, even after death, as white as a sheet. To say "he turned of a dirty yellowish white," would be nearer the truth, though the expression might be thought somewhat inelegant. Poets and others speak of golden hair and silvery locks; but human hair, though it may be bright, glistening, and so on, never reflects light in the manner peculiar to metals. Numerous examples of the same kind will occur to every one. If, therefore, we meet in ancient authors with expressions relating to colour which seem exaggerated and out of place, we must make some allowance for the tendency shown by men at all times to compare one beautiful object with another beautiful object or one terrific object with another terrific object, without regard to exact literal truth. Generally speaking, I think we may safely say that no terms denoting colour, where-ever met with, are to be considered strictly correct and appropriate unless they are referred to some fixed and known standard. The errors due to actual blindness need hardly, I think, be taken into consideration, since the hues which the colour-blind are unable to distinguish lie so far apart as to make it difficult for any one with

normal eyes to conceive the possibility of so great a defect.

The brilliant tints exhibited by the decaying foliage of the trees in this neighbourhood in the course of the autumn, forming a chromatic display such as those living near manufacturing towns have few opportunities of witnessing, have led me to think a little on the cause of the formation of these colours and its possible connexion with chlorophyll, on the chemistry of which I have lately been making some experiments.

The colour of the leaves of plants is a phenomenon which is probably never quite stationary at any period of their development. When lying rolled up in the leaf-bud they are, like underground shoots and other parts of plants that have not been exposed to light, almost white; nevertheless they already contain a colouring-matter called *etiolin,* the alcoholic solution of which is yellow and shows absorption-bands similar to those of chlorophyll. Whether this etiolin on the leaf unfolding passes over into chlorophyll and whether it continues to be formed during the further stage of development is not known. All we know is that when the leaf expands it immediately becomes green from the formation of chlorophyll. It is, however, evident that more than one colouring-matter is formed after exposure of the leaves of plants to light. The colour due to chlorophyll alone is probably seen in its purest state in the tender exquisite green of the young beech leaf or blade of corn. Other leaves, such as those of the oak, before attaining maturity have a decidedly yellow tinge, due, it is supposed, to the presence of an unusual proportion of phylloxanthin, the yellow colouring-matter always accompanying chlorophyll. Indeed the lively contrast of tints seen in the foliage of the woods in early spring, the

yellowish hue of the oak, and the pure green of the beech and larch relieving the sombre colour of the fir tree and the yew, affords one of the most pleasing sights of that delighful season. In early summer the young shoots of some trees, such as the oak, the sycamore, and the thorn, as well as the young leaves near the summit of each shoot, are tinged of a lively red, passing by degrees into the green of the mature leaves. The fruit-wings of the sycamore are for many weeks in the summer similarly tinted; and the effect of the pink blush gradually shading off into the pale green of the wing-tips is one that painters might introduce with advantage into their pictures of still life. This red colour is said to be due to erythrophyll, the colouring-matter formed in some leaves in the autumn; but whether the substance is in both cases really the same may be doubted. At the height of summer the foliage of trees displays a uniform green tint of varying depth; but it is probable that at this season the chlorophyll has already undergone a change, and I suspect that the sombre green of some leaves (such as those of the elm) in summer is partly due to a product of decomposition called "modified chlorophyll," which yields solutions of a much less lively colour than the chlorophyll from which it is formed.

The summer stage is succeeded by that of the autumnal fading of foliage, a change so often observed that it needs no description. With the exaggeration so often employed when coloured objects are referred to, people frequently speak of the multitudinous tints of autumn. In reality, however, these colours, not counting the original green, are only four in number, viz. yellow, brown, red, and purple; and of these the last is a dull inconspicuous colour, while red occurs so seldom in our native trees as to add but little to the total effect when our woods and plantations appear in their autumnal clothing. It is to the passing of

the original green into yellow, and from yellow into brown, and the various shades and tints so produced that the effect is in the main due. The yellow coloration is most distinctly seen in the chestnut and the elm. In the latter the gradual tinging of the deep green with yellow produces a peculiarly beautiful effect; and when the change is complete, and the whole tree (to use one of those figurative expressions to which one is so prone) is arrayed in a garb of gold, the appearance when first seen is almost startling. Arrived at this stage the leaves mostly fall, but retain their yellow hue for a short time only, the colour under the influence of air and moisture rapidly becoming brown, though they remain yellow if quickly dried. The oak and the beech keep their leaves after the yellow stage is passed, and the rich reddish-brown they then exhibit forms a distinct feature in the autumnal landscape. Young beech trees, as every one knows, remain clothed with brown leaves during the winter, and only lose them on the unfolding of the fresh leaves in spring.

The leaves of some of our native plants, such as the wild cherry, the currant, the bramble, and various species of sorrel, turn of a lively red in autumn; but this coloration intensified to a positive scarlet is more distinctly seen in some of the exotics which have been introduced into our gardens, such as the Virginian creeper and the Azalea. A mixture of red and yellow is rarely observed on the same leaf. It is a singular circumstance that the leaves of the oak and sycamore, the young shoots of which are so often tinged of a lively red, do not turn red in fading, but yellow, from which it may be inferred that the process of decay in leaves does not lead back to the same stage at which that of development commenced.

The autumnal purple coloration of leaves is met with in a few native plants, notably the briony, the privet, and the

dogwood. It imparts a dingy hue to the leaves, and is therefore not much noticed. I observed a case of its occurrence last summer, which I had not previously seen mentioned. Passing through a field of corn, which was then nearly ripe, I saw a number of plants by the sides of the path with blades distinctly purple. On closer observation it was evident that in all cases where this coloration occurred the ears of corn had been cut off before ripening, the act probably of idle passers-by, those plants which remained uninjured having become yellow as usual. I inferred that it was the injury sustained by the plant and the arrest of its main function, that of the development of seed, that had led to the formation of some purple substance, not seen during the process of natural decay. I made some experiments on this purple colouring-matter; but all I can say about it is, that it belongs to the same class as the red and yellow colouring-matters of faded leaves. I anticipated the possibility of its being identical with a product of the decomposition of chlorophyll, crystallizing in purple needles, which I had discovered in the course of my investigation; but I was disappointed in my expectation.

As regards the nature of the colouring-matters to which the various colours of faded leaves are due, opinions vary. It is generally supposed that they are formed from chlorophyll by some process of decomposition, probably of oxidation; and nothing can be more natural than this supposition. On exposure of the leaf to light and air, after its vital functions have ceased, the green colour due to chlorophyll gradually disappears and is succeeded by red, yellow, or purple. Therefore, it is argued, the respective colouring-matters must be derivatives of chlorophyll. Nevertheless this view is open to some objection. As regards, in the first place, the red colouring-matter, since no one has succeeded in obtaining it artificially, it must be formed, if a

derivative of chlorophyll, by some process not purely chemical. I have, indeed, obtained as one of the products of decomposition of chlorophyll a substance crystallizing in red laminæ, having a semimetallic appearance by reflected light; but this substance is entirely distinct from the red colouring-matter of faded leaves. The latter may easily be procured, at least in an impure state, by extracting the reddened leaves of the garden Azalea or the Virginian Creeper with boiling spirit of wine, evaporating the extract at a gentle heat, and treating the residue with water, in which the colouring-matter dissolves. The solution has a fine crimson colour like that of red ink. The colour does not change on the addition of such acids as exert no oxidizing action. Nitric acid gradually turns it yellow. By the addition of alkalies, such as ammonia, its colour changes to a yellowish green, and it has then very much the appearance of an alcoholic solution of chlorophyll; but it does not show either before or after the addition of alkali the least indication of absorption-bands, merely a general darkening of the more refrangible end of the spectrum. With lead acetate it gives a grass-green precipitate. These reactions show that the colouring-matter belongs to the same class as that of the rose and red flowers generally; but in the present state of our knowledge regarding this class of substances, it is quite impossible to say whether any two members belonging to the series are identical or not. One thing, however, is certain, viz. that the red colouring-matter of faded leaves is actually formed during the process of decay; it does not pre-exist in the green leaf, though there can be no doubt that leaves which are naturally more or less red, such as those of the copper beech and various species of colium, contain a ready formed red colouring-matter.

As to the yellow colouring-matter of faded leaves, whether it preexists in the green leaf or is formed from chlorophyll

or some other leaf-constituent, it is not so easy to pronounce a decided opinion. This colouring-matter, called by Berzelius *xanthophyll*, is suppsed by some to be identical with phylloxanthin, the yellow substance which, according to Fremy and others, always accompanies the chlorophyll of green leaves. Of its properties little is known; and that little I find to be more or less incorrect. It is said to be soluble in alcohol and ether, insoluble in water, to turn green with acids, and to show a peculiar absorption-spectrum different to that of chlorophyll. These statements require correction. It is, in fact, soluble in water, but insoluble in ether; it does not turn green with acids; and the absorption-bands which it shows are due to an admixture of chlorophyll, as a few simple experiments are sufficient to show. Having taken some bright yellow elm leaves, I extracted them with boiling spirits of wine, and obtained a greenish yellow liquid, which, after filtration, showed only the dark absorption-band in the red corresponding to band I. of the chlorophyll spectrum. I evaporated the extract in the water-bath, and during evaporation observed a deposit form on the sides of the dish consisting of green fat-like masses. On adding water to the residue a portion dissolved, yielding a golden-yellow liquid, while the fat-like masses remained undissolved. After pouring off the liquid and washing the residue with water, the latter was dissolved in hot alcohol, when it gave a yellowish-green liquid which showed all the absorption-bands of chlorophyll distinctly. The golden-yellow watery solution on the other hand showed no trace of absorption-bands, merely a general darkening of the blue end of the spectrum. Its colour was evidently due to a yellow colouring-matter contained in it. It gave an abundant yellow precipitate with lead acetate, and a dark green precipitate with ferric chloride. It also contained a considerable quantity of tannin, since it yielded

a thick curdy precipitate with gelatine and an abundant deposit on the addition of a mineral acid. On again evaporating the solution in the water-bath some decomposition evidently took place; for on adding water to the residue a quantity of matter in the form of brown powder remained undissolved. It is almost certain that it is the same process of decomposition going on in the yellow leaf on exposure to air and moisture that causes the colour to change to brown. I think it probable that the process is one of oxidation, and that it affects the tannin of the leaf or the solution rather than the yellow colouring-matter; for it is well known that watery solutions of tannin undergo decomposition, accompanied by change of colour from light to dark, on exposure to air, especially when the solutions are hot. This simple experiment shows that the yellow colour of faded elm leaves is due partly, perhaps chiefly, to a yellow colouring-matter soluble in water, partly to a yellowish green substance consisting essentially of chlorophyll. The former is, in my opinion, the true xanthophyll.

In order to gain, if possible, a little more insight into the process whereby the colour of green leaves changes to yellow, I took an alcoholic extract of fresh grass, which was of the usual bright green colour, and exposed it in a window to the action of the sun and air. After some days' exposure it had undergone the well known and frequently described transformation; *i. e.* the bright green colour had changed to a greenish yellow, and the solution, from being opaque even in thin layers, had become transparent in consequence of the oxidation of the chlorophyll contained in it. Now this liquid, though not quite so yellow as the alcoholic extract of faded elm leaves, was found closely to resemble the latter. On evaporating over the water-bath and adding water to the residue a yellow liquid was obtained, containing a colouring-matter the reactions of which were

similar to those of the substance from elm leaves. No tannin, however, could be detected in the solution; and this may serve to explain the fact that blades of grass, corn, &c. do not ultimately become brown in fading, but remain yellow. The portion of the residue after evaporation left undissolved by water, was green and fatty; and its solution in alcohol showed the absorption-bands due to modified chlorophyll.

These experiments leave the question of the nature and mode of formation of xanthophyll undecided. It may be one and the same substance in all leaves, or it may differ according to the source whence it is derived; it may be formed by the oxidation of chlorophyll, or it may preexist in the green leaf. I am inclined to think that it exists ready formed in the green leaf, but is not then seen on account of the far greater tinctorial powers of the chlorophyll present at the same time, and that it makes its appearance only when the chlorophyll has been decomposed, the sole trace left by the latter being the slight greenish tinge which all faded yellow leaves show more or less. The varying proportion of xanthophyll contained in green leaves would explain the fact, difficult to understand if we suppose it to be derived from chlorophyll, that some leaves assume a deep yellow colour in fading, while others remain of a pale yellow, and others again are almost colourless when they fall.

I will now conclude, hoping that, if I have not communicated any thing strikingly new, I have at least succeeded in affording the Meeting a few moments' amusement.

Bromley, Kent,
10th December, 1881.

VI. *On Differential Resolvents and Partial Differential Resolvents.* By Robert Rawson, Esq., Assoc. I.N.A., Hon. Member of the Society.

Read January 10, 1882.

Arts. 1 to 8 contain the first and second differential resolvents of a cubic, using therein one independent variable only; and from the second differential resolvent the usual root of the cubic is determined.

In arts. 9 to 11 there is the first partial differential resolvent of a general cubic, using therein two independent variables; and from this partial differential resolvent the root of the general cubic is obtained.

Arts. 12 to 14 contain the first partial differential resolvent of a quartic, using therein three independent variables; and from this resolvent the root of the quartic is determined.

1. Let

$$y^3 + ay + m = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be a cubic in which (a) and (m) are functions of x, a variable parameter.

For convenience, $y^1 = \frac{dy}{dx}$, $a^1 = \frac{da}{dx}$, &c., will be used.

Differentiate (1) with respect to x, then

$$y^1 = -\frac{a^1 y + m^1}{3y^2 + a} = Py^2 + Qy + R, \quad . \quad . \quad . \quad (2)$$

where P, Q, R are functions of (a, m) to be determined as follows :—

$$-a^{\text{I}}y-m^{\text{I}}=3Py^4+3Qy^3+(3R+aP)y^2+aQy+aR.$$

Eliminate from this equation y^4 and y^3 by means of the cubic (1), then

$$\begin{aligned}-a^{\text{I}}y-m^{\text{I}}=3P(-ay^2-my)+3Q(-ay-m)&\\ +(3R+aP)y^2+aQy+aR,&\end{aligned}$$

or

$$(3R-2aP)y^2+(a^{\text{I}}-3mP-2aQ)y+m^{\text{I}}+aR-3mQ=0.$$

To satisfy this equation it is necessary and sufficient that

$$3R-2aP=0, \quad \ldots \quad (3)$$
$$a^{\text{I}}-3mP-2aQ=0, \quad \ldots \quad (4)$$
$$m^{\text{I}}+aR-3mQ=0. \quad \ldots \quad (5)$$

From these equations the values of P, Q, R are readily found to be

$$P(4a^3+27m^2)=9ma^{\text{I}}-6am^{\text{I}},$$
$$Q(4a^3+27m^2)=2a^2a^{\text{I}}+9mm^{\text{I}},$$
$$R(4a^3+27m^2)=6ama^{\text{I}}-4a^2m^{\text{I}};$$
$$\therefore \quad 3R=2aP.$$

Substitute these values in (2), then

$$y^{\text{I}}=\frac{9ma^{\text{I}}-6am^{\text{I}}}{4a^3+27m^2}\cdot y^2+\frac{2a^2a^{\text{I}}+9mm^{\text{I}}}{4a^3+27m^2}\cdot y+\frac{6ama^{\text{I}}-4a^2m^{\text{I}}}{4a^3+27m^2} \quad (6)$$

Equation (6) is therefore the first differential resolvent of the cubic (1).

Hence the cubic (1) is the integral of (6), and the value of y which satisfies (6) is then a root of the cubic (1).

2. To find the second differential resolvent of the cubic (1) it will be convenient to write (6) as follows:—

$$y^{\mathrm{I}}+py^2-Qy+\frac{2ap}{3}=0, \quad . \quad . \quad . \quad . \quad (7)$$

where

$$(4a^3+27m^2)p=6am^{\mathrm{I}}-9ma^{\mathrm{I}}.$$

Differentiate (7) with respect to x, then

$$y^{\mathrm{II}}+2pyy^{\mathrm{I}}+p^{\mathrm{I}}y^2-Qy^{\mathrm{I}}-Q^{\mathrm{I}}y+\frac{2a^{\mathrm{I}}p}{3}+\frac{2ap^{\mathrm{I}}}{3}=0. . \quad (8)$$

From (7) there results

$$yy^{\mathrm{I}}=Qy^2-py^3-\frac{2apy}{3}$$

$$=Qy^2+\frac{apy}{3}+mp.$$

Substitute this value in (8), then

$$y^{\mathrm{II}}+\left(2Q+\frac{p^{\mathrm{I}}}{p}\right)py^2+\left(\frac{2ap^2}{3}-Q^{\mathrm{I}}\right)y-Qy^{\mathrm{I}}+\frac{2ap^{\mathrm{I}}}{3}+\frac{2a^{\mathrm{I}}p}{3}$$
$$+2mp^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

From (7) we obtain

$$py^2=Qy-\frac{3ap}{3}-y^{\mathrm{I}}.$$

Then (9) becomes

$$y^{\mathrm{II}}-\left(3Q+\frac{p^{\mathrm{I}}}{p}\right)y^{\mathrm{I}}+\left(\frac{2ap^2}{3}+\frac{Qp^{\mathrm{I}}}{p}+2Q^2-Q^{\mathrm{I}}\right)y+2mp^2$$
$$-\frac{4apQ}{3}+\frac{2pa^{\mathrm{I}}}{3}=0. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

But

$$2mp^2-\frac{4apQ}{3}+\frac{2pa^{\mathrm{I}}}{3}=\frac{2p}{3}(3mp-2aQ+a^{\mathrm{I}})=0.$$

Then (10) may be written

$$y^{\mathrm{II}}-\left(3\mathrm{Q}+\frac{p^{\mathrm{I}}}{p}\right)y^{\mathrm{I}}+\left(\frac{2ap^{2}}{3}+\mathrm{Q}\frac{p^{\mathrm{I}}}{p}+2\mathrm{Q}^{2}-\mathrm{Q}^{\mathrm{I}}\right)y=0. \quad (11)$$

Restore the values of p and Q, then

$$\frac{d^{2}y}{dx^{2}}-\left\{\frac{d}{dx}\log\frac{3ma^{\mathrm{I}}-2am^{\mathrm{I}}}{\sqrt{4a^{3}+27m^{2}}}\right\}\frac{dy}{dx}+\left\{\frac{a^{\mathrm{I}}m^{\mathrm{II}}-m^{\mathrm{I}}a^{\mathrm{II}}}{2am^{\mathrm{I}}-3ma^{\mathrm{I}}}\right.$$

$$\left.+\frac{6am(a^{\mathrm{I}})^{3}-6a(m^{\mathrm{I}})^{3}-6a^{2}m^{\mathrm{I}}(a^{\mathrm{I}})^{2}}{(2am^{\mathrm{I}}-3ma^{\mathrm{I}})(4a^{3}+27m^{2})}\right\}y=0. \quad . \quad . \quad (12)$$

Equation (12) is therefore the second differential resolvent of the cubic (1). The value of y which satisfies (12) is a root of the cubic (1); and either of the three roots of the cubic (1) is a particular solution of the differential equation (12).

3. When (a) is constant, then the first and second differential resolvents respectively become

$$\frac{4a^{3}+27m^{2}}{6am^{\mathrm{I}}}\cdot\frac{dy}{dx}+y^{2}-\frac{3m}{2a}\cdot y+\frac{2a}{3}=0, \quad . \quad . \quad (13)$$

$$\frac{d^{2}y}{dx^{2}}-\left\{\frac{d}{dx}\log.\left(\frac{-2am^{\mathrm{I}}}{\sqrt{4a^{3}+27m^{2}}}\right)\right\}\frac{dy}{dx}-\frac{3(m^{\mathrm{I}})^{2}}{4a^{3}+27m^{2}}\cdot y=0. \quad (14)$$

4. If (m) be such as to satisfy

$$\frac{m^{\mathrm{I}}}{\frac{4a^{3}}{27}+m^{2}}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

then the coefficient of $\frac{dy}{dx}$ in (14) is zero, and it becomes

$$\frac{d^{2}y}{dx^{2}}-\frac{y}{9}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The general solution of (16) is well known to be

$$y=(c_1\epsilon^x)^{\frac{1}{3}}+(c_2\epsilon^{-x})^{\frac{1}{3}}, \quad . \quad . \quad . \quad . \quad (17)$$

where c_1, c_2 are arbitrary constants to be determined. Integrate (15) with respect to x, then

$$m=\frac{\epsilon^x}{2}-\frac{2a^3}{27}\epsilon^{-x}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Hence equation (17) is a root of the cubic

$$y^3+ay+\frac{\epsilon^x}{2}-\frac{2a^3}{27}\epsilon^{-x}=0.. \quad . \quad . \quad . \quad (19)$$

The values of c_1, c_2 are readily determined by substituting the value of y in (17) in (19); they are as follows:—

$$c_1=-\tfrac{1}{2},$$

$$c_2=\frac{2a^3}{27}.$$

Substitute these values in (17), then

$$y=\frac{a}{3}\left(2\epsilon^{-x}\right)^{\frac{1}{3}}-\left(\frac{\epsilon^x}{2}\right)^{\frac{1}{3}} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

is a root of the cubic (19).

5. The values of ϵ^x, ϵ^{-x} in (19) can be determined by means of a quadratic, so as to satisfy the classical cubic

$$y^3+ay+b=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This equation will coincide with (20) if

$$b=\frac{\epsilon^x}{2}-\frac{2a^3}{27}\epsilon^{-x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Then

$$\epsilon^{x} = b - \sqrt{b^2 + \frac{4a^3}{27}}.$$

And

$$\frac{2a^3}{27}\epsilon^{-x} = -\left(\frac{b}{2} + \frac{1}{2}\sqrt{b^2 + \frac{4a^3}{27}}\right).$$

Therefore

$$y = \left\{-\frac{b}{2} + \frac{1}{2}\sqrt{b^2 + \frac{4a^3}{27}}\right\}^{\frac{1}{3}} - \left\{\frac{b}{2} + \frac{1}{2}\sqrt{b^3 + \frac{4a^3}{27}}\right\}^{\frac{1}{3}}. \quad (23)$$

The value of y in (23) agrees with Cardan's formula.

The process of obtaining the first and second differential resolvents is a direct process; and it is only fair to state that it has been gathered entirely from the correspondence with Sir James Cockle, F.R.S. &c., and the Rev. Robert Harley, F.R.S. &c., with whom originated the important invention of differential resolvents of algebraical equations.

I am not sure whether the results in Arts. (4) and (5) have been published by either Sir James Cockle or the Rev. Robert Harley.

It is more than curious that Cardan's formula should be reached as it has been, without the slightest assumption, by means of the second differential resolvent. And the method which has been adopted is, in my opinion, very suggestive in the theory of higher algebraical equations, especially so when several independent variables are made use of.

6. If in (14) we put

$$\frac{-2am'}{\sqrt{4a^3 + 27m^2}} = \frac{2ar}{\sqrt{27}}, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where (r) is another function of x,

then

$$\frac{(m^{1})^2}{4a^3+27m^2}=\frac{r^2}{27}.$$

Hence there results by substitution

$$\frac{d^2y}{dx^2}-\frac{dr}{rdx}\cdot\frac{dy}{dx}-\frac{r^2}{9}\cdot y=0, \quad . \quad . \quad . \quad (25)$$

which is the second differential resolvent of the cubic

$$y^3+ay+\frac{\epsilon^{-\int rdx}}{2}-\frac{2a^3}{27}\epsilon^{\int rdx}. \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The value of (m) is found by integrating (24) and solving algebraically with respect to (m), to be

$$m=\frac{1}{2}\epsilon^{-\int rdx}-\frac{2a^3}{27}\epsilon^{\int rdx}. \quad . \quad . \quad . \quad . \quad (27)$$

7. If y and z are such as to satisfy the equation

$$\frac{dy}{ydx}=\beta z+\frac{dr}{2rdx}, \quad . \quad . \quad . \quad . \quad . \quad (28)$$

substitute this value of y in terms of z in (25), and it becomes

$$\frac{dz}{dx}+\beta z^2=\frac{1}{4\beta}\left\{-2\frac{d}{dx}\left(\frac{dr}{rdx}\right)+\left(\frac{dr}{rdx}\right)^2+\frac{4r^2}{9}\right\}, \quad (29)$$

an equation which is soluble by means of (26) and (28) for all values of the function r.

8. Put $r=Ax^{2n}$, where (A) is constant, then

$$\frac{dz}{dx}+\beta z^2=\frac{n(n+1)}{\beta x^2}+\frac{A^2}{9\beta}x^{4n}, \quad . \quad . \quad . \quad (30)$$

which is soluble by means of (31) and (32) :—

$$\frac{dy}{ydx}+\beta z+\frac{n}{x}; \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$y^3+ay+\frac{1}{2}\epsilon^{-\frac{Ax^{2n+1}}{2n+1}}-\frac{2a^3}{27}\epsilon^{\frac{Ax^{2n+1}}{2n+1}}=0. \quad . \quad . \quad (32)$$

Equation (30) coincides with the Riccatian form in one case only, viz. when the exponent of x is zero.

This property vanquishes all hopes of connecting the solution of Riccati's equation with the roots of a cubic in its present form. This result was communicated to Sir James Cockle, who kindly sent me the following neat solution of (30).

Assume the Riccatian

$$\frac{du}{dt}+w^2=1.$$

Change the independent and dependent variables t and u, for x and z, by the equations

$$3(2n+1)t=A^{\frac{1}{2}}x^{2n+1};$$

$$3\beta z=A^{\frac{1}{2}}x^{2n}w-\frac{3n}{x}.$$

Then the above Riccatian on reduction coincides with (30).

The solution of a general cubic by a partial differential resolvent with two independent variables :—

9. Let

$$V^3+3aV^2+3RV+3S=0 \quad . \quad . \quad . \quad (33)$$

be a general cubic where (a) is constant, and R, S functions of x, y.

Differentiate (33) with respect to x, y respectively, then

$$(V^2+2aV+R)\frac{dV}{dx}+\frac{dR}{dx}.V+\frac{dS}{dx}=0; \quad . \quad . \quad (34)$$

$$(V^2+2aV+R)\frac{dV}{dy}+\frac{dR}{dy}.V+\frac{dS}{dy}=0. \quad . \quad . \quad (35)$$

From these two equations it follows that

$$\left(\frac{d\,\mathrm{R}}{dy}.\mathrm{V}+\frac{d\,\mathrm{S}}{dy}\right)\frac{d\,\mathrm{V}}{dx}=\left(\frac{d\,\mathrm{R}}{dx}.\mathrm{V}+\frac{d\,\mathrm{S}}{dx}\right)\frac{d\,\mathrm{V}}{dy}. \quad . \quad (36)$$

Equation (34), which is a partial differential equation, is the *first partial differential resolvent* of the cubic (33) together with the conditional equation (36).

Hence it follows that the value of V which satisfies (34), and satisfies also the conditional equation (36), is a root of the cubic (33); and each of the roots of the cubic (33) is a solution of the partial differential equations (34), (35), and (36).

10. Since R, S are arbitrary functions of x, y, it remains to determine them so as to satisfy the equations (34), (35), and (36) when

$$\mathrm{V}=x+y+\alpha,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

where α is a constant quantity.

Substitute V as given in (37) in (36), then

$$\left(\frac{d\,\mathrm{R}}{dy}-\frac{d\,\mathrm{R}}{dx}\right)(x+y+\alpha)+\frac{d\,\mathrm{S}}{dy}-\frac{d\,\mathrm{S}}{dx}=0.. \quad . \quad (38)$$

Now, if the first term of (38) is a quadratic in terms of x, y, then the second term must be a quadratic also. This readily suggests the following equation, viz.

$$\frac{d\,\mathrm{R}}{dy}-\frac{d\,\mathrm{R}}{dx}=\beta(x-y). \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Substitute this value in (38), and it becomes

$$\frac{d\mathrm{S}}{dy}-\frac{d\mathrm{S}}{dx}=\beta y^2-\beta x^2+\alpha\beta y-\alpha\beta x. \quad . \quad . \quad . \quad (40)$$

The integrals of (39) and (40) are

$$\mathrm{R}=\beta xy+\mathrm{C},. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

$$3\mathrm{S}=\beta y^3+\beta x^3-3\alpha\beta xy+3\mathrm{C}_1;. \quad . \quad . \quad . \quad (42)$$

where C, C_1 are independent of x and y.

The values of β, α, C are determined from (34), and are as follows, $\alpha=-a$; $\beta=-1$, and $C=a^2$.

The constant $3C_1=a^3$ is found by substituting V, R, S in the cubic (33). Hence,

$$V=x+y-a \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

is a root of the general cubic.

$$V^3+3aV^2+3(a^2-xy)V-x^3-y^3-3axy+a^3=0 \quad . \quad . \quad (44)$$

The remaining two roots must be obtained from

$$V^2+(x+y+2a)V+x^2+y^2-xy+ax+ay+a^2=0 \quad . \quad . \quad (45)$$

11. The values of x, y in (44) can be found by a quadratic so as to make (44) coincide with the classical cubic

$$V^3+3aV^2+3bV+c=0 \quad . \quad . \quad . \quad . \quad . \quad (46)$$

For this purpose the equations

$$a^2-xy=b, \quad . \quad . \quad . \quad . \quad . \quad (47)$$

$$x^3+y^3+3axy=a^3-c \quad . \quad . \quad . \quad . \quad (48)$$

will be necessary.

From these two equations there results

$$x^3=\frac{3ab-2a^3-c}{2}+\frac{1}{2}\sqrt{(3ab-2a^3-c)^2-4(a^2-b)^3},$$

$$y^3=\frac{3ab-2a^3-c}{2}-\frac{1}{2}\sqrt{(3ab-2a^3-c)^2-4(a^2-b)^3}.$$

The solution of a quartic by a partial differential resolvent with three independent variables :—

12. Let

$$V^4+tV^2+RV+S=0 \quad . \quad . \quad . \quad . \quad . \quad (49)$$

be a quartic in which t, R, S are functions of the three variables x, y, z.

Differentiate (49) with respect to x, y, z respectively, then

$$(4V^2+2tV+R)\frac{dV}{dx}+\frac{dt}{dx}.V^2+\frac{dR}{dx}.V+\frac{dS}{dx}=0, \quad . \quad (50)$$

$$(4V^2+2tV+R)\frac{dV}{dy}+\frac{dt}{dy}.V^2+\frac{dR}{dy}.V+\frac{dS}{dy}=0, \quad . \quad (51)$$

$$(4V^2+2tV+R)\frac{dV}{dz}+\frac{dt}{dz}.V^2+\frac{dR}{dz}.V+\frac{dS}{dz}=0. \quad . \quad (52)$$

From (50) and (51), from (51) and (52) there results

$$\left(\frac{dt}{dy}.V^2+\frac{dR}{dy}.V+\frac{dS}{dy}\right)\frac{dV}{dx}=\left(\frac{dt}{dx}.V^2+\frac{dR}{dx}.V+\frac{dS}{dx}\right)\frac{dV}{dy}, (53)$$

$$\left(\frac{dt}{dy}.V^2+\frac{dR}{dy}.V+\frac{dS}{dy}\right)\frac{dV}{dz}=\left(\frac{dt}{dz}.V^2+\frac{dR}{dz}.V+\frac{dS}{dz}\right)\frac{dV}{dy}. (54)$$

Equation (50) is the first partial differential resolvent of the quartic (49) together with the two conditional equations (53) and (54).

The value of V, in functions of x, y, z, which satisfies (50), and satisfies also the conditional equations (53), (54), is a root of the quartic (49); and a root of the quartic (49) is a solution of each of the equations 50 to 54.

13. With a view, therefore, to obtain a solution of (53), (54), it will be necessary to try a few simple assumptions of the forms of t, R, S, which are suggested by the equations themselves. Put

$$\frac{dt}{dy}=1; \quad \frac{dt}{dx}=1; \quad \frac{dR}{dy}-\frac{dR}{dx}=0. \quad . \quad . \quad (55)$$

By integrating these equations, then

$$t=x+y+z_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

$$R=(y-x)z_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

where z_1, z_2 are functions of z only.

The above assumption, viz. (55), necessarily implies that S is a function of x, y only, or $\frac{dS}{dz}=0$.

Substitute the above values in (53), (54), then

$$\left(V^2+z_2V+\frac{dS}{dy}\right)\frac{dV}{dx}=\left(V^2-z_2V+\frac{dS}{dx}\right)\frac{dV}{dy}, \quad . \quad (58)$$

$$\left(V^2+z_2V+\frac{dS}{dy}\right)\frac{dV}{dz}=\left(\frac{dz_1}{dz}.V^2+\frac{dz_2}{dz}(y-x)V\right)\frac{dV}{dy}. (59)$$

If, then, the roots of the quartic (49) be such as to satisfy

$$V^2+z_2V+\frac{dS}{dy}=0, \quad . \quad . \quad . \quad . \quad . \quad (60)$$

$$V^2-z_2V+\frac{dS}{dx}=0, \quad . \quad . \quad . \quad . \quad . \quad (61)$$

these equations will satisfy (58), (59), provided that

$$\frac{dz_1}{dz}=2z_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

$$\frac{dS}{dy}-\frac{dS}{dx}=\frac{dz_2}{dz}(y-x). \quad . \quad . \quad . \quad . \quad (63)$$

Since S is a function of x, y only, then

$$\frac{dz_2}{dz}=-1, \quad \text{or } z_2=-z.$$

Integrating (62), (63) we have

$$z_1=-z^2,$$
$$S=xy.$$

Substitute these values in (60), (61), then

$$V^2-zV+x=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

$$V^2+zV+y=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

And

$$t=x+y-z^2,$$
$$R=z(x-y).$$

14. The values of V in (64), (65) which satisfy (50) also, are the roots of the quartic

$$V^4+(x+y-z^2)V^2+z(x-y)V+xy=0. \quad . \quad (66)$$

Equation (66) will coincide with the quartic

$$V^4+aV^2\pm bV+c=0, \quad . \quad . \quad . \quad . \quad . \quad (67)$$

if x, y, z are determined from

$$x+y-z^2=a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

$$z(x-y)=\pm b, \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

$$xy=-c. \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

These values depend on the well-known cubic

$$z^6+2az^4+(a^2-4c)z^2-b^2=0.$$

See Todhunter's 'Theory of Equations,' p. 112, 2nd ed.

Havant, Sept. 1881.

Postscript.—Equation (6) is made linear by the relation

$$y=\frac{4a^3+27m^2}{6am^1-9ma^1}\cdot\frac{dz}{zdx}, \quad . \quad . \quad . \quad . \quad (71)$$

which gives

$$\frac{d^2z}{dx^2}+\frac{d}{dx}\log\left(\frac{(4a^3+27m^2)^{\frac{5}{6}}}{2am^1-3ma^1}\right)\frac{dz}{dx}+6a\left(\frac{2am^1-3ma^1}{4a^3+27m^2}\right)^2 z=0. \quad (72)$$

VII. *An Analysis of the recorded Diurnal Ranges of Magnetic Declination, with the view of ascertaining if these are composed of Inequalities which exhibit a true Periodicity.* By Balfour Stewart, M.A., LL.D., F.R.S., Professor of Natural Philosophy at the Owens College, Manchester, and William Dodgson, Esq.

Read March 8th, 1881.

1. It is well known that Professor Rudolph Wolf has endeavoured to render observations of sun-spots made at different times and by different observers comparable with each other, and has thus formed a list exhibiting approximately the relative sun-spot activity for each year. This list extends back into the seventeenth century, and is unquestionably of much value. Nevertheless it must be borne in mind that we possess no sun-spot data sufficiently accurate for a discussion in a complete manner of questions relating to solar periodicity before the time when Schwabe had finally matured his system of solar observations, which was not until the year 1832.

We have, however, a much longer series of the Diurnal ranges of Magnetic Declination. Now these are already well known to follow very closely all the variations of sun-spot frequency, being greatest when there are most, and least when there are fewest spots; and it may even be imagined that such ranges give us a better estimate of true solar activity than that which can be derived from the direct measurement of spotted areas.

The long-period inequalities of the diurnal range of magnetic declination are thus, we may imagine, precisely

those of solar activity, so that to analyze the former is probably equivalent to analyzing the latter.

2. Our method of analysis is not new. The peculiar treatment which we adopted in a previous communication (Proc. Roy. Soc. May 29, 1879), and which had the advantage of enabling us to analyze with comparatively little trouble a very long series of observations, is here unnecessary. The system which we have here pursued is, in fact, that pursued by Baxendell and other astronomers with observations of variable stars; and it will be best understood by means of an example. We shall select this example from a preliminary paper by one of us, in which the long-period inequality in rainfall was roughly analyzed as well as these very declination-ranges which we are here treating in a more laborious and accurate manner (Proc. Lit. & Phil. Soc. Manchester, Feb. 24, 1880).

The Paris rainfall-records arranged so as to exhibit an 8-years series give us the following numbers :—

51·4, 47·5, 45·7, 48·7, 51·1, 49·8, 46·5, 47·2,

the mean being 48·5. From these we obtain the following series of differences :—

+2·9, −1·0, −2·8, +0·2, +2·6, +1·3, −2·0, −1·3.

If we now add these together *without respect of sign,* and divide by their number (8), we obtain 1·76 as the mean departure from the mean of the whole. It is, in fine, mean departures of this kind for periods of varying lengths which we wish to obtain in our present communication.

3. The observations at our disposal are those which have been used by Prof. Elias Loomis in his "Comparison of the mean daily Range of the Magnetic declination with the Extent of the Black Spots on the Surface of the Sun" (American Journal of Science and Arts, vol. l. No. cxlix.). They are as follows :—

(1)	Observations at Paris by Cassini,		1784–1788 ;
(2)	„	„ London, by Gilpin,	1786–1805 ;
(3)	„	„ London by Beaufoy,	1813–1820 ;
(4)	„	„ Paris by Arago,	1821–1831 ;
(5)	„	„ Göttingen,	1834–1840 ;
(6)	„	„ Munich,	1841–1850 ;
(7)	„	„ Prague,	1839–1876 ;

4. These observations are recorded as monthly means of diurnal declination-range; and it is first necessary to multiply each by a certain factor differing for each month of the year, on account of the well-known annual inequality of declination-range.

This factor is determined for each place by means of the observations taken at that place. For London this factor is taken from the Kew observations. For Paris it is taken from Arago's observations; while for Göttingen, Munich, and Prague the factors are taken respectively from the whole body of observations at those places.

5. In the next place, it is necessary to multiply these observations so corrected by a factor in order to bring them to the Prague standard. We have applied for this purpose precisely the same corrections as those made by Prof. Loomis. They are as follows :—The observations at

Paris,	1784–88,	are diminished	by	one fifth;
London,	1786–1820,	„	„	one tenth;
Paris,	1821–31,	„	„	one fourth;
Göttingen,	1834–40,	„	„	one fifteenth;
Munich,	1841–50,	„	„	$\frac{1}{22\cdot47}$;

this last factor has been determined by us.

6. The compound factor embodying both of these corrections is exhibited in the following Table :—

TABLE I.

Exhibiting Compound Factor for correcting the Observations.

Name of Station.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
Paris (1784–88)......	1·186	1·047	0·713	0·627	0·692	0·667	0·714	0·702	0·735	0·739	1·026	1·337
Paris (1821–31)......	1·112	0·982	0·669	0·588	0·649	0·625	0·669	0·658	0·689	0·693	0·962	1·253
London (1786–1820)	1·337	1·087	0·829	0·711	0·793	0·745	0·769	0·744	0·820	0·951	1·168	1·475
Göttingen (1834–40)	1·585	1·328	0·849	0·658	0·721	0·745	0·767	0·731	0·810	0·991	1·551	2·071
Munich (1841–50)...	1·961	1·355	0·866	0·676	0·722	0·726	0·774	0·741	0·817	1·014	1·672	2·152
Prague (1839–76), one factor only.	1·592	1·239	0·977	0·814	0·795	0·732	0·754	0·789	1·009	1·145	1·488	1·794

TABLE II.

Reduced Diurnal Declination-Ranges.

Station.	Year.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
Paris.	1784.	10·40	9·39	7·65	7·00	10·54	7·63	7·16	7·59	8·61	6·73	6·40	5·96
	1785.	7·71	7·53	7·00	6·89	10·03	7·83	8·38	9·47	10·55	9·03	9·50	10·48
Paris and London.	1786.	12·18	11·18	10·99	10·70	10·12	9·93	11·29	11·04	11·82	13·49	11·04	12·84
	1787.	15·30	13·57	12·71	11·21	12·54	12·21	13·93	13·81	12·21	11·75	12·69	13·19
	1788.	11·86	11·15	12·03	12·86	11·69	12·00	10·62	8·22	9·99	11·41	12·06	13·93
London.	1789.	(13·73)	(13·53)	(13·33)	(13·14)	(12·94)	12·74	(11·94)	(11·15)	(10·35)	(9·56)	(8·76)	7·97
	1790.	11·23	(11·33)	(11·43)	(11·53)	(11·64)	(11·74)	11·84	(11·38)	(10·92)	(10·46)	(10·01)	(9·55)
	1791.	9·09	(9·61)	(10·14)	10·67	(11·01)	(11·35)	11·69	(10·94)	(10·20)	(9·45)	(8·71)	(7·96)
	1792.	7·22	(7·69)	(8·17)	(8·64)	9·12	(9·23)	(9·34)	9·45	9·10	8·46	4·32	4·57
	1793.	5·75	5·00	7·05	8·32	8·25	9·39	9·61	9·00	8·04	6·65	4·44	5·61
	1794.	6·01	(6·45)	(6·88)	(7·31)	(7·75)	(8·18)	8·61	7·29	6·89	(7·10)	(7·30)	(7·51)
	1795.	(7·71)	(7·92)	8·13	(7·75)	(7·38)	7·00	6·46	(6·35)	6·23	(5·93)	(5·62)	5·31
	1796.	(5·48)	(5·64)	5·80	(6·30)	(6·80)	7·30	7·77	(7·29)	6·81	(6·95)	(7·09)	7·23
	1797.	(6·87)	(6·50)	6·14	(6·97)	(7·81)	8·64	7·77	(7·00)	6·23	(6·62)	(7·00)	7·38
	1798.	(6·91)	(6·44)	5·97	(6·76)	(7·55)	8·34	7·69	(7·70)	7·71	(6·47)	(5·23)	3·98
	1799.	(4·73)	(5·47)	6·22	(6·83)	(7·44)	8·05	8·00	(7·20)	6·40	(5·94)	(5·48)	5·02
	1800.	(5·25)	(5·49)	5·72	(6·52)	(7·32)	8·12	7·07	(6·69)	6·32	(5·74)	(5·15)	4·57
	1801.	(5·48)	(6·39)	7·30	(7·55)	(7·80)	8·05	7·92	(8·10)	8·28	(6·75)	(5·22)	3·69
	1802.	(5·08)	(6·48)	7·88	(7·91)	(7·94)	7·97	9·46	(8·38)	7·30	(6·74)	(6·17)	5·61
	1803.	(7·00)	(8·39)	9·79	(9·65)	(9·52)	9·39	10·07	(8·93)	7·79	(6·67)	(5·55)	4·43

	1804.	(5·71)	(7·00)	8·29	(8·33)	(8·3 8)	8·42	8·00	(7·81)	7·63	(6·91)	(6·18)	5·46
	1805.	(5·88)	(6·30)	6·72	(7·58)	(8·45)	9·31	8·00	(7·81)	7·63	(7·35)	(7·07)	6·79
	1806.												
	1807.												
	1808.												
	1809.												
	1810.												
	1811.												
	1812.												
London.	1813.				8·46	7·03	7·23	6·56	5·63	5·55	6·84	3·15	3·64
	1814.	5·31	6·66	7·17	7·82	7·15	7·17	7·88	7·13	7·16	7·24	5·00	3·79
	1815.	4·58	7·25	7·34	8·31	8·34	8·28	7·61	6·02	5·78			
	1816.												
	1817.	......	......	......	9·14	8·13	8·25	8·36	8·61	7·03	9·19	7·13	5·87
	1818.	7·91	7·04	6·90	7·63	7·55	8·49	8·14	8·40	8·92	7·63	9·67	6·30
	1819.	5·61	6·12	6·97	7·50	6·88	7·61	7·44	7·64	7·46	6·35	7·03	5·68
	1820.	5·08	6·31	7·27	7·00	7·48	7·02	7·94	7·13	7·56	8·31	6·13	5·19
Paris.	1821.	9·62	7·31	7·60	7·25	6·92	6·60	7·02	7·02	6·44	5·21	5·93	5·00
	1822.	5·72	6·62	6·73	6·66	7·03	7·02	6·89	6·92	6·46	6·65	6·52	5·09
	1823.	6·19	4·63	6·49	6·99	6·67	6·16	6·82	6·52	6·37	5·52	5·14	4·42
	1824.	4·94	4·68	6·23	5·96	6·01	6·46	6·08	6·48	6·19	7·13	6·70	6·25
	1825.	6·06	8·07	7·62	7·59	7·22	6·94	8·32	8·26	7·30	6·53	5·78	6·00
	1826.	6·51	7·89	8·20	7·39	7·28	7·47	7·21	6·98	7·67	7·56	6·89	5·83
	1827.	6·88	8·09	7·97	9·48	8·52	7·86	8·01	8·64	8·69	9·16	8·57	9·67
	1828.	8·42	10·39	8·75	8·67	8·78	9·71	9·55	9·20	8·39	6·55	5·92	8·95
	1829.	12·76	11·12	8·01	8·38	8·33	10·82	9·49	9·11	10·32	11·61	15·08	12·89
	1830.	9·93	8·22	9·43	8·66	10·28	7·99	7·63	7·88	9·20	11·11	10·50	12·92
	1831.	13·15	8·76	6·18	9·54	9·09	8·21						
	1832.												
	1833.												

TABLE II. (*continued*).

Station.	Year.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
Göttingen.	1834.	(7·70)	(7·70)	(7·70)	7·20	7·77	7·68	7·94	7·59	7·74	7·41	8·42	7·52
	1835.	6·94	7·21	8·60	8·92	9·41	8·76	8·11	9·31	9·00	9·61	11·74	11·03
	1836.	8·78	10·37	10·40	11·28	11·97	11·39	11·91	11·73	11·95	12·86	11·71	13·57
	1837.	16·13	11·79	11·31	12·39	11·02	11·65	11·78	9·86	9·95	11·04	11·53	11·14
	1838.	12·96	12·89	12·49	11·71	12·28	12·98	11·51	11·89	12·16	10·73	9·51	9·82
Göttingen & Prague.	1839.	10·39	9·93	9·94	9·20	9·35	9·95	10·42	11·89	10·64	11·21	11·39	8·69
	1840.	9·53	11·55	10·22	10·36	9·20	8·77	9·04	8·56	9·57	9·19	8·36	12·00
Munich and Prague.	1841.	8·68	8·45	8·23	7·79	8·14	8·45	7·46	7·34	7·61	7·54	7·14	7·78
	1842.	7·10	6·65	7·11	7·13	7·24	7·01	6·34	6·86	6·41	7·21	6·53	5·98
	1843.	6·72	6·30	6·06	6·58	6·66	7·33	7·18	7·02	7·54	6·89	5·54	6·47
	1844.	5·10	4·32	6·53	6·24	6·22	6·57	6·79	6·83	6·58	7·47	6·94	6·46
	1845.	5·15	6·21	7·27	7·55	8·05	7·94	7·25	7·95	7·12	6·83	6·85	8·08
	1846.	6·09	7·53	8·13	8·04	9·12	8·28	8·89	8·60	8·06	7·96	8·77	6·98
	1847.	6·29	8·01	8·46	8·47	8·89	8·37	8·52	9·89	10·13	10·72	12·06	11·28
	1848.	12·23	11·20	10·34	9·29	10·42	10·28	11·41	11·64	11·17	10·33	9·63	9·59
	1849.	13·28	11·31	11·40	11·14	10·27	10·09	9·95	8·83	8·93	9·29	9·42	8·55
	1850.	11·41	10·68	10·12	9·60	10·43	10·26	9·93	9·28	9·95	9·45	9·80	7·91
	1851.	10·17	7·74	7·00	7·91	8·56	9·06	8·80	7·84	8·63	8·89	8·27	10·84
	1852.	10·06	12·54	9·81	8·62	8·44	8·83	7·86	8·00	8·94	8·97	9·94	8·70
	1853.	7·88	6·52	8·62	7·08	6·46	7·41	8·00	6·99	7·17	7·09	7·84	8·69
	1854.	7·80	10·75	7·79	8·23	8·43	7·39	7·95	7·58	6·40	7·17	5·73	7·32
	1855.	7·56	8·18	7·73	7·35	7·09	7·05	7·34	6·91	6·60	8·04	7·04	5·33
	1856.	5·48	6·95	5·31	6·63	6·42	6·99	6·79	7·30	7·34	6·89	6·50	5·15
	1857.	6·19	6·43	5·83	6·47	7·08	7·59	7·82	7·58	7·26	8·02	8·72	7·27
	1858.	7·91	9·06	7·66	8·66	3·96	3·90	5·24	8·46	10·67	11·46	8·75	7·34

Prague.	1859.	5·52	7·75	10·29	11·51	10·65	9·84	9·92	10·45	13·74	12·16	10·59	14·37
	1860.	9·31	10·06	11·77	9·41	10·88	11·43	11·27	12·06	9·63	9·37	7·28	7·68
	1861.	7·98	10·47	8·47	9·86	9·39	9·46	8·72	9·15	8·55	7·48	9·25	10·19
	1862.	7·55	6·16	6·06	7·71	7·49	9·85	9·53	9·83	7·54	10·92	9·37	8·56
	1863.	11·02	9·58	9·04	8·80	9·70	8·15	8·07	7·82	8·05	8·59	8·91	9·92
	1864.	8·42	7·12	9·15	7·39	8·25	8·71	7·71	7·94	6·94	8·44	8·58	6·91
	1865.	9·04	8·81	9·89	7·83	8·48	7·59	7·13	7·54	8·43	7·03	8·00	4·20
	1866.	9·89	11·41	7·04	7·60	7·30	7·16	7·15	5·87	6·78	6·16	9·48	5·71
	1867.	7·00	6·47	7·29	7·09	7·01	7·24	7·68	6·92	6·70	5·93	6·44	6·87
	1868.	7·37	7·05	8·36	9·14	7·46	7·35	7·99	8·20	7·96	7·76	8·45	9·65
	1869.	7·88	8·78	9·52	8·78	8·71	10·71	9·93	8·71	9·57	9·24	7·86	7·84
	1870.	9·63	8·34	11·03	11·69	12·24	11·11	11·75	11·54	11·41	10·85	12·05	11·81
	1871.	9·98	13·04	11·09	12·02	10·87	11·35	11·42	12·25	10·60	10·02	13·33	10·93
	1872.	11·54	8·70	9·68	10·89	10·61	10·08	10·58	10·87	10·76	9·39	11·63	12·67
	1873.	12·93	8·87	9·45	9·83	8·56	7·33	8·58	8·53	8·58	8·59	8·30	8·65
	1874.	10·36	9·23	7·86	7·83	8·01	7·63	7·92	6·95	7·85	7·13	7·35	5·49
	1875.	3·92	7·00	6·56	7·15	7·31	7·24	6·61	6·95	6·58	5·86	5·24	5·62
	1876.	6·08	5·25	6·04	6·00	6·11	6·97	7·42	6·32	5·79	6·71	5·61	6·10

7. When the corrections of Table I. have been applied, we obtain results indicating in minutes of arc the mean diurnal declination-range, and which we may suppose to be reasonably well freed from the influence of yearly variation as well as from that due to locality. It is proper to remark that the Prague ranges have been deduced from the observations taken at the hours 18, 22, 2, 10, as these hours were common to the whole series with the exception of 1853. It seems possible that the result of confining ourselves to these hours has been to make the ranges for *Prague alone* somewhat too small. We do not, however, regard our results as any thing more than a first approximation. It is likewise necessary to remark that for those months during which we have two sets of observations we have taken the mean of both. The monthly means are recorded in Table II., those in brackets being interpolated values.

8. Our next operation has been to transform the monthly values of Table II. into quarterly ones, four of these going to each year. This is done by adding the monthly values together in threes. We have not divided the sums so obtained by three, preferring to adopt the scale of 20″, which we, of course, obtain by adding three monthly values together without subsequent division. This scale will therefore henceforth be adopted in this communication.

9. The next point is to arrange these quarterly values into series of various lengths, and find for each series the mean departure from the mean of the whole, as already described in art. 2 of this paper.

As we are now dealing with quarterly values, a series arranged for eleven years will contain 44 terms, one for twelve years 48 terms, and so on. In some few cases we have deduced the mean departures by means of yearly

values, not quarterly ones. Such are indicated by brackets.

TABLE III.—Mean Departures from Mean of whole.

Period in years.	Mean departure.	Period in years.	Mean departure.
3	0·6655	11	(1·60)
4	0·6462	$11\frac{1}{2}$	1·8535
5	0·7558	$11\frac{3}{4}$	1·9793
$5\frac{3}{4}$	0·6567	12	2·1373
6	1·0242	$12\frac{1}{4}$	1·9607
$6\frac{1}{4}$	0·6979	$12\frac{1}{2}$	1·5406
7	(0·89)	13	(0·68)
8	(0·61)	14	(1·12)
9	(0·47)	15	(1·26)
$9\frac{3}{4}$	1·5235	$15\frac{3}{4}$	1·6866
10	2·1454	16	1·7109
$10\frac{1}{4}$	2·5428	$16\frac{1}{4}$	1·7948
$10\frac{1}{2}$	2·6271	$16\frac{1}{2}$	1·6830
$10\frac{3}{4}$	2·2705	17	(1·35)

10. It appears from Table III. that there are marked indications of inequalities corresponding in period to $10\frac{1}{2}$ and to 12 years. As these periods are near together, we have eliminated the influence of the 12-yearly inequality upon the results for $10\frac{1}{4}$, $10\frac{1}{2}$, and $10\frac{3}{4}$ years. This gives us the following corrected values :—

Period in years.	Mean departure.
$10\frac{1}{4}$	2·3163
$10\frac{1}{2}$	2·5629
$10\frac{3}{4}$	2·2872

The period of the inequality is thus manifestly $10\frac{1}{2}$ years. In the next place we have eliminated the influence of this $10\frac{1}{2}$-yearly inequality so determined upon the results for $11\frac{3}{4}$, 12, and $12\frac{1}{4}$ years, and have thus obtained the following corrected values :—

Period in years.	Mean departure.
$11\frac{3}{4}$	2·0556
12	2·1490
$12\frac{1}{4}$	1·9538

The values of the $10\frac{1}{2}$- and the 12-yearly inequalities freed by this means from the influence of each other are found in Table IV. (p. 66 *seqq.*).

We next eliminated the influence of these two inequalities from the whole series of quarterly observations, and with the values so corrected obtained the indications given in Table III. of an inequality at $16\frac{1}{4}$ years. The value of this inequality is found in Table IV. Besides these three inequalities, there appear to be traces of an inequality corresponding to 6 years. This, however, will be incorporated in that whose period is 12 years, and ought not, therefore, to be separately considered.

11. Having thus determined three inequalities with periods of $10\frac{1}{2}$, 12, and $16\frac{1}{4}$ years, it becomes a point of interest to know whether we can reproduce the observed results reasonably well by their means. The united effect of these three inequalities is given in Table IV. col. 5. The result is that the dates of observed maxima and minima, as well as observed ranges, are well reproduced by adding together these three inequalities. Nevertheless there is the following peculiarity remaining behind. Let there be two curved lines, the upper line being drawn through the various *observed* maximum points, and the lower through the various *observed* minimum points of declination-range. The spaces between these two lines may be taken to denote the *observed capacities for declination-range* corresponding to the various years.

Again, in like manner, let there be two other curved lines, the upper line being drawn through the various *calculated* maximum points and the lower through the various

calculated minimum points of declination-range, while the spaces between the two lines may be taken to denote the *calculated capacities for declination-range.*

It will be found that the observed and calculated range-capacities agree well together; nevertheless there is this peculiarity about the observed results. There appears to be an exaltation of the observed zero or base-line when the range is great, and a depression of the same when the range is small.

12. This is a phenomenon which we might expect from the facts connected with smaller periods. Let us take the eleven-yearly period (as it is called) whether for sun-spots or declination-ranges. It is well known that the short-period oscillations (those of perhaps two or three months) are much more pronounced in both cases at epochs of maximum than at epochs of minimum. In fine, we are here imagining something analogous to musical beats, in virtue of which more especially the two periods $10\frac{1}{2}$ and 12 years alternately produce a state of exaltation and depression. Indeed we may even imagine that these two inequalities of $10\frac{1}{2}$ and 12 years themselves represent the beat-period of some shorter disturbance.

13. We have therefore endeavoured to give a variable zero to our calculated ranges, on the supposition which follows—namely, observed zero $= x + y \times$ calculated range.

Now the observed zero for each year may be obtained by taking a point midway between the two observed curves, while the calculated ranges may be obtained from the second set of curves. We shall thus get a series of equations of the above type, one for each year; and by making use of these we may obtain the values of x and y. By this means we have found $x = 19{\cdot}07$, $y = 0{\cdot}54$.

14. Having thus obtained our three periods and a means of calculating the position of the variable zero, we may

now attempt a reproduction of the past, and perhaps even a forecast of the future, as far as declination-range is concerned.

The calculations necessary for this are given in Table IV.; and a graphical comparison of the observed and calculated results will be found in the diagram (p. 69).

The mean difference between observed and calculated results is 39″; but there is reason for supposing that ultimately a still nearer approximation will be obtained.

On the whole this is a good agreement; and the result appears to us in favour of the hypothesis that the long-period inequalities of declination-range exhibit a true periodicity. Nevertheless, owing to paucity of material, this analysis is by no means complete.

It will be seen that while there is, according to our calculation, the probability of a maximum about 1884, it will not be so high as that in 1871, and we shall afterwards enter on a period of diminished magnetic, and probably diminished solar action. Something of the kind was suspected by the late Mr. Broun.

TABLE IV.*

Year.	12·0.	10·5.	16·25.	Sum.	Calculated range.	Range × 0·54.	Add 19·07.	Declination.		
								Calculated.	Observed.	Difference.
1784.	−3·98	+0·30	−0·72	−4·40	19·7	10·65	29·72	25·32	23·76	+1·56
1785.	2·26	2·66	0·63	0·23	19·7	10·65	29·72	29·49	26·10	3·39
1786.	0·09	4·38	+0·86	+5·15	18·8	10·16	29·23	34·38	34·15	0·23
1787.	+2·72	3·87	3·16	9·75	18·2	9·84	28·91	38·66	38·78	−0·12
1788.	4.11	1·63	1·98	7·72	17·1	9·24	28·31	36·03	34·45	+1·58
1789.	2·80	0·27	3·19	6·26	16·1	8·70	27·77	34·03	34·78	−0·75
1790.	1·52	−1·97	2·33	1·88	14·9	8·05	27·12	29·00	33·26	4·26

* In this table the previous sign is supposed to be repeated until a new one is given.

TABLE IV. (*continued*).

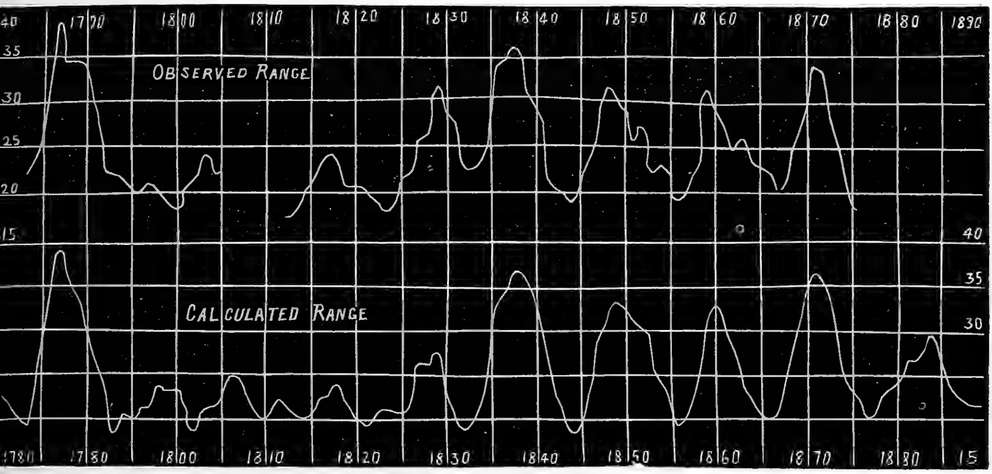
Year.	12·0.	10·5.	16·25.	Sum.	Calculated range.	Range × 0·54.	Add 19·07.	Declination. Calculated.	Declination. Observed.	Declination. Difference.
1791.	+1·17	−3·07	+1·08	−0·82	13·7	7·40	26·47	25·65	30·20	−4·55
1792.	0·17	3·94	−0·39	4·16	12·7	6·86	25·93	21·77	23·83	2·06
1793.	−0·79	3·28	2·00	6·07	11·5	6·22	25·29	19·22	21·78	2·56
1794.	1·70	0·97	1·34	4·01	10·5	5·68	24·75	20·74	21·82	1·08
1795.	3·67	+1·38	1·62	3·91	9·2	4·97	24·04	20·13	20·45	0·32
1796.	3·98	3·85	1·40	1·53	8·0	4·32	23·39	21·86	20·11	+1·75
1797.	2·26	4·39	1·00	+1·13	7·0	3·78	22·85	21·72	21·23	0·49
1798.	0·09	2·70	0·83	1·78	6·0	3·24	22·31	24·09	20·19	3·90
1799.	+2·72	1·09	1·98	1·83	5·3	2·86	21·93	23·76	19·19	4·57
1800.	4·11	−1·01	1·04	2·06	4·7	2·54	21·61	23·67	18·49	5·18
1801.	2·80	2·52	0·88	−0·60	4·3	2·32	21·39	20·79	20·63	0·16
1802.	1·52	3·64	+0·15	1·97	4·2	2·27	21·34	19·37	21·73	−2·36
1803.	1·17	3·82	2·92	+0·27	4·1	2·22	21·29	21·56	24·29	2·73
1804.	0·17	2·30	2·46	0·33	4·3	2·32	21·39	21·72	22·03	0·31
1805.	−0·79	+0·30	3·08	2·59	4·7	2·54	21·61	24·20	22·22	+1·98
1806.	1·70	2·66	2·36	3·32	5·1	2·76	21·83	25·15		
1807.	3·67	4·38	1·54	2·25	5·7	3·08	22·15	24·40		
1808.	3·98	3·87	−0·15	−0·26	6·0	3·24	22·31	22·05		
1809.	2·26	1·63	1·45	2·08	6·2	3·35	22·42	20·34		
1810.	0·09	0·27	2·19	2·01	6·4	3·46	22·53	20·52		
1811.	+2·72	−1·97	0·74	+0·01	6·4	3·46	22·53	22·54		
1812.	4·11	3·07	2·07	−1·03	6·5	3·51	22·58	21·55		
1813.	2·80	3·94	1·05	2·19	6·6	3·57	22·64	20·45	18·03	2·42
1814.	1·52	3·28	0·58	2·34	6·3	3·41	22·48	20·14	19·87	0·27
1815.	1·17	0·97	1·72	1·52	5·6	3·03	22·10	20·58	21·17	−0·59
1816.	0·17	+1·38	0·82	+0·73	5·0	2·70	21·77	22·50		
1817.	−0·79	3·85	1·44	1·62	4·5	2·43	21·50	23·12	23·90	0·78
1818.	1·70	4·39	0·00	2·69	4·0	2·16	21·23	23·92	23·64	+0·28
1819.	3·67	2·70	+2·32	1·35	3·4	1·84	20·91	22·26	20·57	1·69
1820.	3·98	1·09	2·83	−0·06	3·1	1·68	20·75	20·69	20·56	0·13
1821.	2·26	−1·01	2·56	0·71	2·6	1·41	20·48	19·77	20·48	−0·71
1822.	0·09	2·52	2·48	0·13	2·5	1·35	20·42	20·29	19·58	+0·71
1823.	+2·72	3·64	1·57	+0·65	2·8	1·51	20·58	21·23	17·98	3·25
1824.	4·11	3·82	0·14	0·43	3·4	1·84	20·91	21·34	18·28	3·06
1825.	2·80	2·30	−1·10	−0·60	4·4	2·38	21·45	20·85	21·42	−0·57
1826.	1·52	+0·30	2·11	0·29	5·9	3·19	22·26	21·97	21·72	+0·25
1827.	1·17	2·66	0·75	+3·08	7·3	3·95	23·02	26·10	25·38	0·72
1828.	0·17	4·38	2·50	2·05	8·9	4·81	23·88	25·93	25·82	0·11
1829.	−0·79	3·87	0·40	2·68	9·9	5·35	24·42	27·10	31·98	−4·88
1830.	1·70	1·63	0·65	−0·72	11·1	6·00	25·07	24·35	28·44	4·09
1831.	3·67	0·27	1·48	4·88	12·4	6·70	25·77	20·89	27·44	6·55
1832.	3·98	−1·97	1·44	7·39	13·4	7·24	26·31	18·92		
1833.	2·26	3·07	1·40	6·73	14·5	7·84	26·91	20·18		
1834.	0·09	3·94	0·19	4·22	15·1	8·16	27·23	23·01	23·09	0·08
1835.	+2·72	3·28	+1·91	+1·35	15·8	8·54	27·61	28·96	26·93	+2·03
1836.	4·11	0·97	2·69	5·83	16·4	8·86	27·93	33·76	34·48	−0·72

TABLE IV. (*continued*).

Year.	12·0.	10·5.	16·25.	Sum.	Calculated range.	Range × 0·54.	Add 19·07.	Declination. Calculated.	Declination. Observed.	Declination. Difference.
1837.	+2·80	+1·38	+2·28	+6·46	16·8	9·08	28·15	34·61	34·90	−0·29
1838.	1·52	3·85	2·77	8·14	17·1	9·24	28·31	36·45	35·23	+1·22
1839.	1·17	4·39	1·83	7·39	17·3	9·35	28·42	35·81	30·75	5·06
1840.	0·17	2·70	0·56	3·43	17·3	9·35	28·42	31·85	29·09	2·76
1841.	−0·79	1·09	−0·42	−0·12	17·4	9·40	28·47	28·35	23·65	4·70
1842.	1·70	−1·01	2·45	5·16	17·4	9·40	28·47	23·31	20·39	2·92
1843.	3·67	2·52	0·82	7·01	17·2	9·30	28·37	21·36	20·07	1·29
1844.	3·98	3·64	2·03	9·65	17·1	9·24	28·31	18·66	19·01	−0·35
1845.	2·26	3·82	1·32	7·40	16·8	9·08	28·15	20·75	21·56	0·81
1846.	0·09	2·30	0·69	3·08	16·6	8·97	28·04	24·96	24·11	+0·85
1847.	+2·72	+0·30	0·88	+2·14	16·1	8·70	27·77	29·91	27·77	2·14
1848.	4·11	2·66	2·38	4·39	15·8	8·54	27·61	32·00	31·88	0·12
1849.	2·80	4·38	0·72	6·46	15·0	8·11	27·18	33·64	30·61	3·03
1850.	1·52	3·87	0·63	4·76	14·4	7·78	26·85	31·61	29·70	1·91
1851.	1·17	1·63	+0·86	3·66	13·8	7·46	26·53	30·19	25·93	4·26
1852.	0·17	0·27	3·16	3·60	13·2	7·13	26·20	29·80	27·68	2·12
1853.	−0·79	−1·97	1·98	−0·78	12·9	6·97	26·04	25·26	22·44	2·82
1854.	1·70	3·07	3·19	1·58	12·5	6·76	25·83	24·25	23·13	1·12
1855.	3·67	3·94	2·33	5·28	12·2	6·59	25·66	20·38	21·55	−1·17
1856.	3·98	3·28	1·08	6·18	12·1	6·54	25·61	19·43	19·44	0·01
1857.	2·26	0·97	−0·39	3·62	12·1	6·54	25·61	21·99	21·56	+0·43
1858.	0·09	+1·38	2·00	0·71	12·3	6·65	25·72	25·01	23·27	1·74
1859.	+2·72	3·85	1·34	+5·23	12·5	6·76	25·83	31·06	31·70	−0·64
1860.	4·11	4·39	1·62	6·88	12·9	6·97	26·04	32·92	30·04	+2·88
1861.	2·80	2·70	1·40	4·10	13·2	7·13	26·20	30·30	27·24	3·06
1862.	1·52	1·09	1·00	1·61	13·4	7·24	26·31	27·92	25·39	2·53
1863.	1·17	−1·01	0·83	−0·67	13·8	7·46	26·53	25·86	26·91	−1·05
1864.	0·17	2·52	1·98	4·33	14·2	7·68	26·75	22·42	23·89	1·47
1865.	−0·79	3·64	1·04	5·47	14·5	7·84	26·91	21·44	23·49	2·05
1866.	1·70	3·82	0·88	6·40	14·8	8·00	27·07	20·67	22·89	2·22
1867.	3·67	2·30	+0·15	5·82	15·2	8·22	27·29	21·47	20·66	+0·81
1868.	3·98	+0·30	2·92	0·76	15·5	8·38	27·45	26·69	24·18	2·51
1869.	2·26	2·66	2·46	+2·86	15·8	8·54	27·61	30·47	27·13	3·34
1870.	0·09	4·38	3·08	7·37	16·0	8·65	27·72	35·09	33·36	1·73
1871.	+2·72	3·87	2·36	8·95	16·0	8·65	27·72	36·67	34·22	2·45
1872.	4·11	1·63	1·54	7·28	16·0	8·65	27·72	35·00	31·85	3·15
1873.	2·80	0·27	−0·15	2·92	15·9	8·59	27·66	30·58	27·05	3·53
1874.	1·52	−1·97	1·45	−1·90	15·8	8·54	27·61	25·71	23·40	2·31
1875.	1·17	3·07	2·19	4·09	15·5	8·38	27·45	23·36	19·01	4·35
1876.	0·17	3·94	0·74	4·51	14·9	8·05	27·12	22·45	18·60	3·85
1877.	−0·79	3·28	2·07	6·14	14·2	7·68	26·75	20·61		
1878.	1·70	0·97	1·05	3·72	13·8	7·46	26·53	22·81		
1879.	3·67	+1·38	0·58	2·87	13·1	7·08	26·15	23·28		
1880.	3·98	3·85	1·72	1·85	12·6	6·81	25·88	24·03		
1881.	2·26	4·39	0·82	+1·31	12·0	6·49	25·56	26·87		

TABLE IV. (*continued*).

Year.	12·0.	10·5.	16·25.	Sum.	Calculated range.	Range × 0·54.	Add 19·07.	Declination.		
								Calculated.	Observed.	Difference.
1882.	−0·09	+2·70	−1·44	+1·17	11·3	6·11	25·18	26·35		
1883.	+2·72	1·09	0·00	3·81	10·7	5·78	24·85	28·66		
1884.	4·11	−1·01	+2·32	5·42	9·9	5·35	24·42	29·84		
1885.	2·80	2·52	2·83	3·11	9·2	4·97	24·04	27·15		
1886.	1·52	3·64	2·56	0·44	8·5	4·59	23·66	24·10		
1887.	1·17	3·82	2·48	−0·17	7·7	4·16	23·23	23·06		
1888.	0·17	2·30	1·57	0·56	6·7	3·62	22·69	22·13		
1889.	−0·79	+0·30	0·14	0·35	5·9	3·19	22·26	21·91		
1890.	1·70	2·66	−1·10	0·14	5·3	2·86	21·93	21·79		



VIII. *Colorimetry.*—Part VI.

By James Bottomley, B.A., D.Sc.

Read October 5th, 1880.

On a Theory of Mixed Colours.

In some experiments which I was making on the absorption of light, I needed surfaces of different degrees of whiteness. To obtain such surfaces I mixed black and white powders in various proportions. They were blended by shaking in a bottle and grinding in a mortar. In some cases the mixture was reduced to a flat surface by pressure; in other cases a little water was added so as to form a paint, and with this pieces of cardboard were covered and dried. Although it was not necessary for my purpose that I should quantitatively assign the degree of whiteness in each case, yet such a question occurred to me in the preparation of these powders. In this matter little has, I think, been done. Newton, in his 'Optics,' informs us that he mixed powders of different colours in such proportions as to form a grey; and in later times it has been proposed to estimate the degree of blackness of different bodies by finding the quantity of some white powder which, on admixture, will yield some standard tint. But no one has, I think, ever considered the law of intensity of colour when we mix a colour with white or black. Such a question is interesting to the artist, to the colour-mixer, and to the chemist. The artist in water-colour, if he wishes to reduce the strength of any pigment, can do so by additional water; but the painter in oil or distemper,

if he wishes to accomplish a similar object, has to mix with some opaque white. This, then is the matter which I first propose to consider. A coloured powder and a white powder are intimately blended; what is the law of the intensity of colour? As a typical case, I take the mixture of black and white, because it was such a mixture that suggested this inquiry. In what follows I suppose the powders to consist of indefinitely small particles which do not exert any chemical action on each other. Suppose that we take a mass of some white substance and add to it a small quantity of some black substance; then we shall take away some portion of its whiteness. If w denote the whiteness lost and W the initial whiteness, then the remaining whiteness would he $W-w$. If now we add another unit of the black, it might at first sight appear that the remaining whiteness would be $W-2w$, and that, after the addition of n units, the remaining whiteness would be $W-nw$. For some experiments which I was making, I had prepared eight grey tints by mixing $BaSO_4$ with carbon, the quantity of $BaSO_4$ being 10 grams in each case and the carbon increasing from 0·006 gram to 0·048 gram. The difference in tint between the successive mixtures seemed to diminish more rapidly than seemed consistent with such a law of diminution of whiteness. The difference between the seventh and eighth mixtures was almost inappreciable; but, according to the foregoing supposition, the difference between successive pairs should be the same.

On considering the matter further, I was led to the following train of reasoning:—If we take equal masses of white of different intensities, and to each add the same bulk of black, then in each case the whiteness lost will be a constant fraction of the initial whiteness. Suppose M to be the mass of the white, and W_0 its initial whiteness; let m be the mass of the black. After mixing with the

white it will destroy some fraction of its whiteness; let this fraction be n. Then the remaining whiteness will be $W_0 - W_0 n$; this we may denote by W_1; and the mass will be $M + m$.

Suppose we repeat the addition of the black, the proportion being as before, $M : m$. If x denote the black to be added, we shall have the proportion

$$M + m : M :: x : m;$$

whence

$$x = \frac{(M+m)m}{M}.$$

After this second mixture the whiteness will be $W_1 - W_1 n$; this we may denote by W_2; it may also be written $W_0(1-n)^2$ by substitution for W_1, or still more briefly $W_0 R^2$ when $R = (1-n)$.

Let the operation be repeated a third time, the proportion of the white mass to the black being still $M : m$. After the second mixture the mass became $\frac{(M+m)^2}{M}$. So, if x denote the quantity of black to be added, we shall have

$$\frac{(M+m)^2}{M} : M :: x : m.$$

Whence

$$x = m + \frac{2m^2}{M} + \frac{m^3}{M^2}.$$

The mass will now become

$$\frac{(M+m)^3}{M^2}.$$

If W_3 denote the whiteness, we shall have

$$W_3 = W_2 - W_2 n = W_0 R^2 (1-n) = W_0 R^3.$$

If we continue the operation n times, then from the above law, if W_n denote the remaining whiteness, we shall have

$$W_n = W_0 R^n.$$

Also the mass will be

$$\frac{(M+m)^n}{M^{n-1}}.$$

I also used an independent method of reasoning. Suppose we have a white area A, then the quantity of white light given off in any direction, say normal to the surface, will be proportional to A; so that, if W_0 denote the white light, we may write $W_0 = \mu A$. Suppose now a great number of black points to fall on this surface, being equally distributed. Then the surface will appear to the eye of a grey tint; but grey and white are quantities of the same kind and are therefore comparable—what we call grey being a white of diminished intensity. Suppose a to be the area occupied by the black points; then $A - a$ will be the uncovered white area, and the quantity of white light given off by this will be $\mu(A-a)$; moreover this quantity of white light will appear uniformly diffused over the surface. If we denote it by W_1 we shall then have

$$W_1 = \mu(A-a) = \mu A\left(1 - \frac{a}{A}\right) = W_0 R,$$

where R is written for $1 - \frac{a}{A}$. Now suppose a second series of black spots to fall on the surface. It might at first sight appear that the remaining white area would be $A - 2a$; but on consideration this did not seem necessarily the case, for manifestly it supposes that the particles distribute themselves with some bias; that is, they prefer to fall upon a white surface; but suppose that they have no

such bias, and that they will as readily fall upon a black as upon a white surface. Now the surface on which they fall is grey or a mixture of black and white. So we have this question in the distribution of the second black area (consisting of innumerable detached points):—How much falls on the black surface, and how much on the unoccupied white area? Let p be the portion that falls on the black surface, and q the portion that falls on the white; now what will be the ratio of p to q? If we suppose the second series of black points to be fairly distributed, the portion which falls on the black surface will be to the portion which falls on the white as the areas of those surfaces, so that

$$\frac{p}{q}=\frac{a}{A-a}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

also

$$p+q=a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the remaining white area will be $A-a-q$. From (1) and (2) we have

$$\frac{qa}{A-a}=a-q,$$

whence

$$q=\frac{a(A-a)}{A},$$

and for the remaining white area we have the expression

$$A-a-\frac{a(A-a)}{A}=A\left(1-\frac{a}{A}\right)^2=AR^2,$$

and the quantity of white light given off will be μAR^2; also this quantity of light will appear equally distributed over the whole surface; hence, if W_2 denote this white light, we may write

$$W_2=\mu AR^2=W_0R^2.$$

If we allow a third series of black points to fall upon the surface and to be equally distributed, the remaining white area will be AR^3, and if W_3 denote the quantity of white light,

$$W_3 = W_0 R^3.$$

If we suppose the operation to be repeated n times, the expression for the remaining white light will be $W_0 R^n$. Hence the ratio of the initial to the final whiteness would be $\frac{1}{R^n}$. Both trains of reasoning concur in giving a similar expression for the intensity of the whiteness. In some papers which I have contributed to this Society I have pointed out that the law expressing the intensity of transmitted light when we dissolve Q units of colouring-matter in a transparent medium, is of the form $\Sigma \alpha \kappa^Q$. Hence we have this curious result: when the intensity of an opaque white is diminished by mixture with an opaque black, the mathematical expression for the intensity of the whiteness is of the same form as if we had dissolved the black in a transparent medium and surveyed a white area through it. In the foregoing reasoning I have supposed the particles, after admixture, to distribute themselves without bias. It becomes a question of much interest, when we mix particles of heterogeneous matter, is this always the case? Under some circumstances they may be brought within the sphere of molecular attractions, and these may have some influence in the distribution; in other words it is possible to conceive that the particles will distribute themselves with some bias. Here, again, it seems to me that so far from such speculations on the intensity of colour of mixtures being fruitless, they may even extend the application of colorimetry; for while experimental agreement with the theory would strengthen the theory, even negation

would have its value; and an investigation into the departures from the law might lead to interesting results, and give some insight into the operation of those molecular forces which separately elude observation, but whose joint effect must necessarily have some influence in determining the intensity of colour.

Some of the above reasoning applies to the case of turbid liquids; and I was led to the conclusion that a carbon diffusion would behave with regard to the extinction of light in the same manner that it would do if the carbon were actually in solution. The only difference in the reasoning is, that the different series of carbon points, instead of falling on one section, are distributed through a series of circular sections of the containing cylinder, these sections being parallel to the external white surface. As I have shown in another paper, the results of the experiments agree very well with the theory. So far I have considered the intensity of the residual whiteness when we mix black with white. The same course of reasoning might be applied to determine the residual colour when we mix black with any colour. Suppose, for instance, we take a mass M of yellow, and let the initial yellow be Yo. Then, if we mix with it a mass m of black, we shall remove some fraction of the yellow; let this be denoted by nYo, so that the residual yellow is $Yo(1-n)$, or YoR. After n repetitions according to the proportions laid down in the case of black and white, the intensity of the residual yellow will be YoR^n.

Another problem is the mixture of white with some opaque colour, red for instance. Suppose we start with a white area A, then the quantity of white light given off normally may be denoted by μA. Suppose now a great number of red points to fall upon this surface and to be equally diffused, so that the eye does not perceive detached

red and white points, but a surface uniformly tinted of a light red colour. Let a be the area occupied by the red points; then the quantity of red light we may denote by $\mu_1 a$, and the uncovered white area will be $A-a$, and the quantity of white light given off will be $\mu(A-a)$. Suppose now a second series of red points to fall upon the surface, and that they distribute themselves without bias, and also that there is no chemical action. Then the uncovered white area will be $\frac{(A-a)^2}{A}$, and the red area will be $A-\frac{(A-a)^2}{A}$. Hence after the second operation the light given off will consist of white light $\frac{\mu(A-a)^2}{A}$, and of red light $\mu_1\left\{A-\frac{(A-a)^2}{A}\right\}$, or, as we may write them, μAR^2 and $\mu_1 A(1-R^2)$, where $R=1-\frac{a}{A}$. If the operation be repeated n times, the residual whiteness will become μAR^n and the redness will become $\mu_1 A(1-R^n)$. If n becomes infinite, the whiteness vanishes and the red becomes $\mu_1 A$, being the red light that would be given off if we supposed the surface to be covered with red points only.

A method for experimentally testing the foregoing theory relative to the intensity of the residual whiteness after admixture with a perfect black, would be as follows:—Take three surfaces of different degrees of whiteness (A, B, C), due to admixture with p, q, r units of black; look at the surfaces through some fluid containing in solution some soluble black substance, adjust the columns so that the intensity of the transmitted light shall be the same. Suppose first we compare A and B. Let t and t' be the lengths of the columns. Then

$$W_0 R^p k^t = W_0 R^q k^{t'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now compare B with C, whence, if θ and θ' be the lengths of the columns, we shall have

$$W_0 R^q k^\theta = W_0 R^r k^{\theta'} . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From equation (3) we have

$$R = k^{\frac{t'-t}{p-q}};$$

and from equation (4) we have

$$R = k^{\frac{\theta'-\theta}{q-r}}.$$

But these two values of R ought to be the same; so we ought to have the equation

$$\frac{t'-t}{p-q} = \frac{\theta'-\theta}{q-r}.$$

The different tints I used consisted of $BaSO_4$ and lampblack. Tint A consisted of lampblack 0·012 gram, $BaSO_4$ 10 grams; tint B contained twice the above quantity of lampblack to the same quantity of $BaSO_4$; and tint C contained four times the quantity of lampblack to the same quantity of $BaSO_4$. The absorbing-medium I used consisted of water containing a minute quantity of lampblack in suspension. This, as I have before shown, behaves nearly the same with regard to the absorption of light as if the carbon were in solution.

A comparison of tint A with tint B gave $R = k^{\cdot 3}$. Tint A compared with tint C gave $R = k^{\cdot 346}$. Tint B compared with C gave $R = k^{\cdot 4}$. Inasmuch as k is a fraction, these three values of R will not differ much. The experimental inquiry is difficult; and the following defect is likely to have some influence on the result. The grey powders were

mixed with a few drops of water and pieces of cardboard covered with the paint so obtained, and then dried. But if we take an intimate mixture of two powders, and make it into a paint with oil or water, the gravity of the two powders being different, and the fluid medium imparting a certain degree of mobility to the particles, there will be a tendency for the lightest powder to come to the surface. I have sometimes noticed, with regard to the tablets prepared as above, that portions which had been rubbed seemed perceptibly lighter than the undisturbed portion; possibly this may be due to a slight excess of carbon on the upper surface.

I also compared tint A with another consisting of 0·4003 gram carbon to 10 grams $BaSO_4$; thus the quantity of carbon is a considerable multiple of that contained in tint A. The value of R got from the comparison differed considerably from a value of R which I got by comparing A with B. At first I thought this was due to a failure in the theory; after some time it occurred to me that the conditions of my experiments were not the same as the conditions of the theory. In the theory I had supposed that the white was mixed with a perfect black; in the experiments the white had been mixed with grey.

Those surfaces which are popularly known as black are in reality not black but grey. If we take such a surface, whether of black velvet or lampblack, and hold an opaque object before it so as to intercept a portion of the incident light, a shadow will be found on the surface; but it is evident that a perfect black is incapable of receiving a shadow. Also, when I looked at a surface of lampblack through the colorimeter (consisting of a glass cylinder covered with black cloth, except a small aperture at the bottom) the lampblack surface appeared grey. The formula for the intensity of the residual whiteness, if we

mix with grey, will not be the same as if we mixed with black. The formula will have to be altered as follows:—Suppose W_0 the initial whiteness, after the addition of perfect black the residual whiteness will be W_0R^n; but if the material added be grey, we must give back a quantity of white, which will be some fraction of the whiteness lost. Suppose p to be this fraction; also the whiteness lost will be $W_0 - W_0R^n$, so that the quantity of white to be restored will be $W_0\,p(1-R^n)$, and the total whiteness will be $W_0R^n(1-p)+W_0\,p$. If we suppose n to become infinite, the whiteness becomes $W_0\,p$, being the whiteness or greyness of the so-called black body. We might also have deduced the formula as follows:—Take a white area A, then the quantity of white light given off we may denote by μA. Now let a series of grey points fall upon this; let a be the area of the spots; then the quantity of white light given off by this we may denote by $\mu_1 a$, the uncovered white area will be $A-a$, and the quantity of white light given off by this will be $\mu(A-a)$; therefore the whole quantity of white light will be $\mu(A-a)+\mu_1 a$, or $\mu AR+\mu_1 a$ if R be written for $1-\frac{a}{A}$. If we suppose another series of grey points distributed over the surface, the uncovered white area will be AR^2, and the surface covered by the grey points will be $A-AR^2$; so that the quantity of light will be $\mu AR^2+\mu_1(A-AR^2)$. If the operation be repeated n times, the expression for the residual whiteness will be $\mu AR^n+\mu_1(A-AR^n)$, which may be written in the form $\mu A[R^n(1-p)+p]$, where $p=\frac{\mu_1}{\mu}$; also $\mu A=W_0$, the initial whiteness; so that the expression is equivalent to the one previously given.

On the Theory of Engraving.

Another subject of interest in Colorimetry is the theory of engraving, which I think has never been considered. In this art various shades of grey are given to white surfaces by aggregations of lines or dots, giving rise to line, mezzo-tint, and other varieties of engraving. If the tint be produced by lines, it may be estimated as follows:— Take a white square area A and rule it with parallel lines. The quantity of white light given off initially we may denote by μA. Let b be the breadth of one of these lines and l its length; also suppose that there are n of these lines; then the white area uncovered will be $A-nbl$. Suppose n to become very great and b very small, so that we no longer have the impression of black lines on a white surface, but see a uniform grey surface; then the expression for its degree of whiteness will be $\mu(A-nbl)$ or $W_0(1-nr)$ if r denote $\frac{bl}{A}$.

Sometimes an engraver, instead of using parallel lines only, crosses the lines (cross hatching). With a given number of lines the tint will not be the same if he draws them all parallel, and if half are drawn at right angles to the others. Suppose we have $2n$ lines. If drawn parallel, the degree of whiteness will be $W_0(1-2nr)$; but let n of these lines be drawn perpendicular to the remaining n. Take the case of one of these perpendiculars: it will intersect one of the first series in a square whose area is b^2; and as it is cut by n lines, the sum of these will be nb^2; the additional white area blotted out by this line will be $lb-nb^2$; and since there are n such lines, the total area they blacken will be $n(lb-nb^2)$. Hence the remaining white area will be $A-nlb-(nlb-nb^2)$, which may be written $A(1-nr)^2$, since $l^2=A$. If W_1 denote the white-

ness when the lines are parallel, and W_2 when they cross, we shall have

$$\frac{W_1}{W_2} = \frac{1-2nr}{(1-nr)^2}.$$

In both cases the lines are supposed to be so thin as to be individually imperceptible.

Again, suppose an engraver to cover a square white area with black circular spots which touch one another, the spots being very numerous and individually imperceptible, so that we receive the impression of a grey surface. If beyond this stage he exercise his skill in diminishing the area of the spots indefinitely and increasing their number so that they still fulfil the condition of touching, it will make no difference in the intensity of the tint; for the area of the spots is always the same, and equal to that of the circle that can be inscribed in a square.

IX. *On the Motion of Developable Cylinders.*

By James Bottomley, B.A., D.Sc., F.C.S.

Read before the Physical and Mathematical Section, March 29, 1881.

I have not seen in any works on Dynamics the following problem treated :—A perfectly flexible surface is rolled up so as to form a cylinder; the external edge is fixed, and the cylinder is allowed to roll down an inclined plane under the action of gravity: to determine the motion. As

examples we may take rolls of tape or ribbon. I suppose the lamina to have thickness, but this thickness to be indefinitely small. Approximately the motions may be determined as follows :—Let a section of the cylinder be made by a vertical plane through its centre of gravity. Take the fixed point as origin. The external forces acting on the mass are $g \sin a$ and the tension at the fixed point; these act along the plane, which may be taken for the direction of the axis of x; perpendicular to the plane the external forces are $-g \cos a$ and the resistance of the plane; these acting in the direction of the axis of y. Of the whole tension at the fixed point a portion will be due to the unrolled part of the cylinder in contact with the plane, also of the whole resistance of the plane a portion will be due to the unrolled portion. For each particle in contact with the plane $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ vanish. If then T_1 denotes the tension due to the rolling mass and R_1 the resistance, the ordinary equations of motion will give

$$\Sigma m \frac{d^2x}{dt^2} = M_1 g \sin a - T_1, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\Sigma m \frac{d^2y}{dt^2} = -M_1 g \cos a + R_1, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the summation extending to all the particles of the rolling mass. If we suppose it to move as a rigid body, these equations may be written :—

$$M_1 \frac{d^2x}{dt^2} = M_1 g \sin a - T_1;$$

$$M_1 \frac{d^2y}{dt^2} = -M_1 g \cos a + R_1.$$

If we take moments about the centre of the rolling cylinder,

the equation will be

$$\Sigma m\left\{x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right\}=\mathrm{T}_1y.$$

Or, more simply, if we suppose the cylinder to move as a rigid body,

$$\frac{d}{dt}\left(\mathrm{M}_1k^2\frac{d\theta}{dt}\right)=\mathrm{T}_1y, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

M_1k^2 denoting the instantaneous moment of inertia. From (1) and (3) we have the following equation :—

$$\mathrm{M}_1y\frac{d^2x}{dt^2}+\frac{d}{dt}\left(\mathrm{M}_1k^2\frac{d\theta}{dt}\right)=\mathrm{M}_1yg\sin a. \quad . \quad . \quad (4)$$

Also we have the following geometrical equation, since the space x is described by a circle of variable radius :— $dx=yd\theta$. Also the mass in contact with the plane will be $bdcx$, where c denotes thickness, b and d breadth and density. If we regard the rolling portion as a circular cylinder, its mass will be $bd\pi y^2$, supposing it to have the same density as the unrolled portion. Let M be the whole mass, and R the initial radius, then $\mathrm{M}=\pi\mathrm{R}^2bd$. Since the mass is constant, we obtain the equation $\pi y^2=\pi\mathrm{R}^2-cx$; since y and x are the coordinates of the centre of gravity, this equation gives a parabola as its locus. Also $k^2=\frac{y^2}{2}$; hence equation (4) may be written

$$\frac{d^2x}{dt^2}-\frac{c}{2\pi y^2}\left(\frac{dx}{dt}\right)^2=\frac{2}{3}g\sin a.$$

By integration we obtain

$$\left(\frac{dx}{dt}\right)^2=\frac{1}{(\pi\mathrm{R}^2-cx)}\left\{\mathrm{A}-\frac{2g}{3c}\sin a\,(\pi\mathrm{R}^2-cx)^2\right\},$$

A denoting the constant. If this be determined by the supposition that the mass starts from rest, the equation may be written as follows :—

$$\left(\frac{dx}{dt}\right)^2 = \frac{2g}{3c} \sin a \pi R^2 \frac{1 - \left(1 - \frac{cx}{\pi R^2}\right)^2}{1 - \frac{cx}{\pi R^2}};$$

or, if l denote the length of the tape,

$$\left(\frac{dx}{at}\right)^2 = \frac{2}{3} g \sin al \frac{1 - \left(1 - \frac{x}{l}\right)^2}{1 - \frac{x}{l}}.$$

Hence as x approaches to the value l the velocity increases indefinitely. The whole tension at the fixed point at any time will be

$$Mg \sin a - gM_1 \sin a \frac{1}{3} \frac{1 + \left(1 - \frac{x}{l}\right)^2}{\left(1 - \frac{x}{l}\right)^2}.$$

The initial tension will be $\dfrac{Mg \sin a}{3}$.

If the external end of the tape were attached to the contiguous coil so that it could not unroll, the tension at the fixed point would be $Mg \sin a$. If the fastening suddenly gave way, the tension would immediately alter and become only one third of its previous value.

If we suppose the length of the tape to be infinite, then the velocity after describing any finite space will be

$$2\sqrt{\frac{g}{3} \sin a \, . \, x}.$$

If we suppose the motion to take place on a horizontal plane, the equation of motion changes; the differential equation now becomes

$$\frac{d^2x}{dt^2}-\frac{c}{2\pi y^2}\left(\frac{dx}{dt}\right)^2=0.$$

A first integral of this equation will be

$$\left(\frac{dx}{dt}\right)^2=\frac{A}{\pi R^2-cx}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To determine A, suppose the mass set in motion by a blow parallel to the plane so that the initial velocity is V, thus $A=\pi R^2 V^2$.

Substituting, integrating again, and determining the constant we have

$$t=\frac{2l}{3V}\left\{1-\left(1-\frac{x}{l}\right)^{\frac{3}{2}}\right\}. \quad . \quad . \quad . \quad . \quad (2)$$

To unwind the whole length of the tape, this equation would make the time to vary as the length directly and as the initial velocity inversely.

Equation (1) may also be written

$$M_1\left(\frac{dx}{dt}\right)^2=MV^2.$$

Hence, as the mass diminishes the velocity increases, but the kinetic energy in the direction of motion is constant and equal to its initial value. Hence it would seem that none of the energy of the blow is consumed during the rotation of the variable cylinder; once started it would continue of itself. In the rolled-up cylinder there is an amount of potential energy which may be estimated as

follows :—Suppose that originally we had a thin lamina resting on a flat plane; now the amount of work necessary to raise a particle of weight w to the height y is wy; and to raise an aggregate of particles the work will be Σwy or $gM\bar{y}$, where $\bar{y}$ denotes the vertical height of the centre of gravity and M the whole mass. In the rolled-up cylinder this is stored up as potential energy; during the motion it assumes the kinetic form, and would of itself be sufficient to keep up the motion on a smooth plane. In what precedes I have supposed the centre of gravity to lie in the normal to the plane drawn through the point of contact of the cylinder with the plane. This would not be exactly true; on account of the cylinder not being perfectly circular, there will be an extremely small couple due to gravity tending to produce rotation.

If in equation (2) we suppose the length of the tape to be infinite, for the time of motion during any finite length we shall have $t=\frac{x}{v}$.

In the above problem I have supposed the external edge of the tape to be fixed. We may, however, have the internal edge fixed and the external in motion, as in the following problem :—An indefinitely thin lamina is wound round a fixed horizontal cylinder of indefinitely small cross section; to the external edge of the lamina a weight is fixed: to determine the motion of this edge. Suppose we take moments about the fixed axis. Then the expression

$$\Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)$$

for the whole mass may be divided into two parts. For the portion that is still coiled up the angular velocity will be the same for each particle, and equal to the angular velo-

city of the body about the axis. For this portion then we shall have the expression

$$\frac{d}{dt}\left(\mathrm{M}_1 k^2 \frac{d\theta}{dt}\right).$$

For the unwound portion the angular velocity about the axis is not the same for each particle. At any instant y is the same for each particle, also $\frac{dy}{dt}$ is the same for each particle. The expression of moments for this part will take the form

$$\frac{d}{dt} \cdot \frac{dx}{dt}\left\{\frac{cx}{4y}(\mathrm{R}^2+3y^2)\right\}.$$

The geometrical equations will be the same as before. The positive direction of the axis of x is taken vertically downwards. Suppose w, the attached weight, to have n times the mass of the tape, then the equation of motion may be written

$$\frac{d}{dt} \cdot \frac{dx}{dt}\left\{\frac{\pi y 3}{2} + \frac{cx(\mathrm{R}^2+3y^2)}{4y}\right\} = cxgy + gn\pi \mathrm{R}^2 y.$$

The coefficient of $\frac{dx}{dt}$ may also be written

$$\frac{1}{4y}\left(\frac{2\pi^2\mathrm{R}^4 - c^2x^2}{\pi}\right).$$

Writing, for brevity, this in the form $\mathrm{F}(x)$, the equation of motion becomes

$$\frac{d^2x}{dt^2}\mathrm{F}(x) + \mathrm{F}'(x)\left(\frac{dx}{dt}\right)^2 = cxgy + gyn\pi\mathrm{R}^2.$$

The solution of this equation is

$$\left(\frac{dx}{at}\right)^2 = \frac{2}{[\mathrm{F}(x)]^2}\left\{\int gy(cx+n\pi\mathrm{R}^2)\mathrm{F}(x)dx+c\right\}.$$

If we suppose the motion to start from rest, the constant will be o. Performing the integration, the result will be

$$\frac{dx}{at}=\frac{y}{2-\frac{x^2}{l^2}}\sqrt{\frac{8g}{\pi\mathrm{R}^2}}\sqrt{\frac{x^2}{l}+2nx-\frac{nx^3}{3l^2}-\frac{x^4}{4l^3}}.$$

If we suppose the length of the tape to be infinite, the velocity, after describing any finite space x, will be

$$2\sqrt{\frac{ngx}{\pi}}.$$

X. *On the Development of the Common Frog.*

By Thomas Alcock, M.D.

Read before the Microscopical and Natural-History Section, February 13th and April 17th, 1882.

With the intention of making a series of drawings of frog-tadpoles in successive stages of development, I obtained some spawn which the frogs were seen in the act of depositing on the 3rd of April 1881; and on the 5th I made my first observation, commencing a series which extended to a hundred days, namely to the 12th of July. Drawings were made almost daily; and short descriptions of the appearances presented were written at the same

time. The Plates accompanying this paper contain copies of a selection of the drawings; and the descriptions which follow will serve to explain them. The numbers used, agreeing with those on the Plates, are the number of days since the ova were deposited, and give the age of each specimen represented.

Description of Plates.

Plate I.

3. I placed a single ovum in a watch-glass with water, and examined it under the microscope with a one-inch object-glass. The central part was uniformly opaque and of a light brown colour by reflected light; the surface consisted of tubercles of unequal size, most of them rather large; they were slightly rounded, but flattened on the top; and their sides, where they were in contact, were straight, forming angular figures, most frequently six-sided; but in some cases they had three, four, or five sides; and they varied in size, a small one being here and there interposed among larger ones.

4. The surface of the ovum consisted of tubercles about one sixth the size of those seen yesterday; and their number was proportionately increased. They were remarkably angular in shape.

4 (2). Six hours later they had become much smaller and rounded.

5. The surface was marked with an immense number of very small tubercles; and a few slight irregular depressions appeared here and there.

On the 6th and 7th days there was no perceptible change. The ovum continued perfectly opaque and of a light brown colour; and the surface was marked with minute rounded tubercles of uniform size.

8. The surface consisted, as before, of exceedingly small tubercles, but a considerable area in the middle of that side of the sphere which was uppermost was slightly raised and had an abrupt edge, as if pushed up by some force acting within the yelk-mass.

9. The whole mass had assumed a shape which might be compared to the body of a fiddle, with a deep groove along the median line. It was enclosed in a delicate, transparent, spherical sac.

Plate II.

10. The body had lengthened and become narrower; the head was indicated; and on each side of it were three rounded ridges with intervening furrows; a rudiment of the tail was also visible in the form of a short, thick, rounded lobe. Nothing could be seen of the deep furrow down the middle which was so strongly marked yesterday; there was a broad flat process extending from the back of the head to the body, into which it merged.

11. The whole animal was larger; the head was one fourth larger. The body was rather narrower in proportion to its length. But the most remarkable advance was in the tail: this part was much lengthened, and consisted of two rounded ridges with a deep groove between them; and as the groove did not reach the end, the appearance was that of a thick cord forming a long loop.

The animal was seen to move for the first time, the movement consisting of bending the body into a more arched form forwards, or ventrally, and then relaxing it.

12. The body was longer in proportion to the size of the head; the tail was well formed, and there was a prominent ridge in the middle line above; a similar ridge extended along the back.

12 (2). A second drawing was made six hours later.

The animal was seen to be fish-shaped; and the ridge along the back and tail was distinct. On each side of the head were two thick projecting processes, and on the right side a third, small one, in front of the two. The animal frequently bent itself together so that the head and tail met, and then allowed itself to straighten again as much as the confinement of the egg-membrane would allow.

Plate III.

I have daily compared the ovum in the watch-glass, from which all the drawings have been made, with a large number of others kept in a basin, and have found the one I selected to show a fair average of the rate of development. Many tadpoles in the basin have now, on the 13th day, escaped from the egg-capsule; but there are others in every stage of development, and in the centre of the mass are some ova which, though living, still retain their rounded form.

13. The body was much compressed and decidedly fish-shaped. The external gills were beginning to bud out in the form of two undivided processes on each side of the back of the head. The animal frequently curled up its tail and body close to its head and then relaxed itself.

14. The body was perfectly fish-shaped; and there was a projection under the throat, seen when the animal turned about, but not represented in the drawing. The external gills were large and branched, and were two on each side; on the right side the anterior had two, and the posterior three divisions.

14 (2). A second drawing was made six hours later, when the gills were found to be larger; and on the left side the secondary processes were arranged comb-like, or all on one side of the main stem: they were four in number on the anterior, and five on the posterior.

The tadpole in the watch-glass, from which all the drawings have been made, still remained in the egg-sac; but it continually turned itself from side to side, as if the confinement had become irksome.

15. The tadpole was free and straightened to its full length. A side view showed the sucker under the throat in profile, a round depression in the skin near the front of the head (the future nostrils), and the external gills larger and more branched than before. They and the heart were in active use, as the circulation in them was distinctly seen, the current stopping for a moment after each beat of the heart.

The animal had grown altogether larger; and a two-inch object-glass was used instead of the one-inch which had hitherto been employed.

15 (2). Two views of this tadpole were drawn, the natural size.

PLATE IV.

15 (3). A tadpole with unusually large gills was selected from those in the basin, and a drawing made of it. The depressions for the nostrils were well marked; and it was seen in this, as in the previous specimen, that the gills grow out directly from the surface of each side immediately behind the head.

16. A drawing of the tadpole which was examined yesterday was made, of the natural size.

16 (2). The tadpole which has been in the watch-glass from the first was examined under the microscope with the two-inch object-glass. Two drawings of the under surface of the head, showing the mouth and sucker, were made. The upper lip was rather thick and overhanging; the lower lip was everted; both had even edges. The sucker was a large single cavity with a thick, raised rim. It

was wide and rounded in outline behind, but contracted towards the front, until the two sides were concealed under the lower lip. Within it were two plates shaped like a simple leaf, with the bases attached close together in front, and the points apparently free, and directed backwards. The underside of the head, especially the lips and the borders of the sucker, were clothed with large vibratile cilia, which were in vigorous action and caused a strong current from the front towards the hind part of the body. The circulation was perfectly seen in the gills and in the transparent parts of the body. The gills were fully developed; and a slight commencement of the opercular fold was seen in front of their base.

17. A third drawing was made of the head of the tadpole, representing the mouth and sucker in profile.

17 (2). A complete drawing was made of the tadpole in the watch-glass. It represented the animal with the sucker in action. The form of the abdomen was a long oval. The gills were not reduced in size; but the opercular fold slightly overlapped their base. The depressions of the nostrils were deeper; but there was no external appearance of eyes.

Plate V.

18. The tadpole had grown altogether larger. The head was more pointed in front and looked broader behind from the growth of the opercular fold, which encroached on the bases of the gills in front and formed a projecting flap on each side, causing the appearance of a deep indentation between the head and body when the animal was viewed from above. The gills were darker in colour and less transparent, from the deposition of more black pigment; and though still large, they had a slightly shrunk appearance, their whole surface being corrugated. The

circulation in them was clearly seen. There was no appearance of eyes.

19. The head appeared much more formed, and frog-like in shape. The eyes were distinctly seen; but they were covered by the ordinary skin. The gills had shrunk a little; and they were darker-coloured, with much black pigment; but the circulation could be seen.

19 (2). A tadpole was taken from the basin and killed in sherry and water. A drawing was made of the under surface of the body, showing the large mouth with thick fleshy lips, and a little behind it a pair of suckers. They were comma-shaped, with raised walls, and were joined behind by a slightly raised curved line, the remains of the posterior wall of the original single sucker. The opercular fold closely resembled the gill-cover of fishes; the gills were rather large; and in what might be called (comparing them to arms) the axilla was a double or treble row of small rounded bodies like beads, the first noted appearance of the internal gills.

Plate VI.

A tadpole which was taken from the basin as soon as it was hatched, and had since been kept in a watch-glass with water, was killed in sherry and examined. It was slightly distorted, but not spoiled. Two drawings were made—one of the upper, the other of the lower surface.

20. The upper view showed nothing not previously noted, except that the shape of the head was more decided, as if the skull had advanced in development. The eyes appeared, but were still covered by the ordinary skin; they were opaque white in the dead specimen.

20 (2). The lower view showed the large open mouth with thick fleshy lips, and a horny sheath on the upper jaw. The two suckers were shown as before, but a little wider apart behind, and the curved line joining them was

scarcely visible. In the axilla on the right side were three distinct rows of short round-ended processes of equal length, which were evidently the internal gills. They were placed in a deep depression behind the roots of the plume-gills. On the left side, in the same position, there was a cluster of short processes of equal length and similar in character to those on the right side; but showing more a side view, their arrangement in rows did not appear.

21. A drawing was made of a tadpole, the natural size.

21 (2). It was then killed and examined under the microscope. A drawing was made of the under surface, which showed the suckers diminished in size, the opercular fold meeting across the chest, and a deep depression in each side at the root of the plume-gills. These were diminished in size; but the circulation was still seen in them.

Plate VII.

22nd. A tadpole was killed and examined. The appearances being exactly the same as those represented in the drawing made on the 21st day, except that the suckers were very slightly smaller, a fresh drawing was not made.

At this time the stock of tadpoles in the basin failed: all that remained in it died, apparently in consequence of the water becoming putrid after one or two dead leaves from the pond had been put into it. The spawn which I received at first was a large mass about a foot in diameter; and the whole of it, except the small quantity already mentioned as kept in the house, was placed in an artificial pond in my garden. The pond is 16 yards long, and 9 yards across at the widest part, it is lined throughout with Portland cement, and is about 2 feet deep.

This spawn remained among some dead leaves in a shallow part where it could be easily seen. It was looked at

from time to time, but not closely examined. The ova on the surface of the mass, however, were seen to be hatched; and the tadpoles had escaped before those deeper in the mass began to develop, so that there was a difference of about a fortnight in the hatching of those at the surface and those in the centre.

At this time, namely the 25th of April, all the tadpoles had escaped, and nothing was seen but the mass of gelatinous matter which had surrounded the ova and still held together. I had frequently looked very carefully round the borders of the pond for tadpoles, but could find none; now, however, on plunging a large basin into the midst of the gelatinous mass, hundreds of tadpoles were at once washed into it with the water, and it was evident that the pond would furnish an abundant supply for further observations.

The water of the pond was at this time so clear that in the sunshine every part of the bottom could be clearly seen. It contained a few water-boatmen and about half a dozen sorts of beetles. The sides were well grown over with confervæ, which formed a rich green pile covered with large air-bubbles.

23rd. I examined several of the pond tadpoles, and found no advance in them from the last drawing. The tadpoles in the pond appeared to be in development slightly behind those which had grown in the house.

24. A tadpole was killed and a drawing made of its lower surface. The gills were still rather large, but were dark-coloured and opaque. There was an appearance as if the opercular fold were preparing to join to the skin of the abdomen, the latter forming a slight raised ridge to meet it. The suckers were smaller than before, but were evidently efficient, as they were in constant use by the living tadpoles, which always remained stationary un-

less disturbed, when they swam actively, but quickly reattached themselves.

25th. About a hundred tadpoles were reserved in the basin and kept in the house for examination.

27. Almost all of them were actively swimming about; only a very few continued to use their suckers.

One was killed, and a magnified drawing made of its lower surface. On the right side the skin had completely closed over the gills, the junction being effected by the skin of the abdomen rising in a sharp edge and meeting the edge of the opercular fold. The condition of the opening at the left side, as represented in the drawing, shows how the closure is effected, beginning at the middle line of the body and extending outwards. The suckers were thin and flat, and had evidently lost their function. The mouth was large and the lips frilled.

28. A drawing was made representing a tadpole, of the natural size.

28 (2). It was then killed, and a drawing made, under the microscope, of the upper surface, showing the head and body. The head was bluntly pointed in front, and broad behind in consequence of the addition of the gill-chambers. The eyes were still covered by the ordinary skin, which, however, began to look thinner and more transparent; in the dead specimen they were opaque white. The body in this, as well as in the two previous specimens, was fuller from side to side, tending to give a depressed figure instead of being compressed as at first; the enlargement appeared to be owing to the development of the abdominal viscera.

Plate VIII.

28 (3). A drawing was made of the under surface of a tadpole. It showed the completion of the chambers for the internal gills, leaving only an opening on the left side. The junction of the opercular fold with the skin of

the abdomen was marked by a slightly indented line crossing from side to side.

29 (May 1st). On examining a tadpole, the skin over the eyes was seen to be transparent; but it contained numerous minute black and gold-coloured spots, like those in the skin of other parts of the body. It was continuous over the eyes : and the eyes themselves were seen to move beneath it; each had a black pupil. The nostrils had a small projecting fold in front of each. Rudiments of the hind limbs were distinctly seen, in the form of a pair of prominent papillæ, one on each side of the vent at the root of the tail. A drawing was made representing the right side with one of the rudimentary limbs. A faint trace of these had been previously seen, as a very slight protuberance on each side, in the specimens examined and drawn two days ago (27).

29 (2). A drawing of a tadpole was made, the natural size.

29 (3). After being killed, it was examined under the microscope and a drawing made, representing the upper surface. The body, instead of being laterally compressed and shaped like an ordinary fish, as it was in the earlier stages, was now broad and depressed, and the head and body were nearly equal in width. The chamber on each side for the internal gills gave much of the apparent breadth to the back of the head; and the greatly increased size of the abdomen indicated rapid growth of its contained viscera. The alimentary canal was certainly in action; for fæcal matter was voided whilst the animal was under examination.

Plate IX.

30. Two tadpoles were drawn, the natural size.

One of them was killed and examined under the microscope. It had a broad rounded body. On the upper sur-

face, the skin was of a light transparent brown colour, spotted with small flakes which had a perfect metallic lustre, exactly like bits of gold leaf; and these appeared to be embedded at different depths in the substance of the skin. Round the borders of the head and body the skin formed a fold which was nearly transparent; and some slight movements of the gills could be seen through it. On the left side, the opening of the gill-chamber was seen slightly projecting; and a current out of it could be detected by the presence of small particles in the water.

On the under side, the suckers were very small and wasted; they were evidently useless. A slightly depressed line passed across the middle of the body, showing the junction of the opercular fold with the skin of the abdomen. In this view the gill-aperture could not be seen from below, but was visible when looked at from above. The coiled form of the intestine was plainly seen through the walls of the abdomen. Small rudiments of hind limbs were present as before. The flakes like gold-leaf in the skin were first noticed three days ago.

31. The tadpole examined today was a fine one. The view of its back under the microscope presented a lovely sight; it was a rich light brown colour, spangled with large flakes of gold. There were many small spots of gold in the skin over the eyes; the eyes themselves moved freely about, and they were rather prominent.

A drawing was made of the ventral surface (31). The mouth was quite an elaborate affair: the lips were white and beautifully scolloped and frilled; within the upper lip was a row of very numerous small dark-brown horny teeth; and within the lower lip were three imperfect rows of similar teeth; inside the lips, at the centre, was a pair of brown horny jaws, the upper one the larger, and closing in front of the lower. The suckers had entirely disap-

peared; and their place was marked by two very small scars, blackened by closely set spots of black pigment. The skin of the whole body was dotted over with very small black spots; and here and there were gold spangles, not so numerous nor so regularly placed on the ventral side as on the back. There was a distinct slightly indented line, marking the union of the opercular fold with the skin of the abdomen; and on approaching the left side it turned backwards and formed the right border of a funnel, at the end of which was the opening of the gill-chamber. The rudiments of the hind limbs were no larger than they were ten days ago.

A few days since, some of the confervæ growing on the sides of the pond was scraped off and put into the basin with the tadpoles to serve for food.

32. Three life-size drawings of tadpoles were made: two of these show different sizes among tadpoles of the same age; the third is a side view, showing the protuberant abdomen.

The tadpoles in the basin still attach themselves to the sides, most frequently near the surface of the water. They seem to have some difficulty in doing so, and they are not very secure when attached; but they pass perhaps half their time thus stationary. Excepting in a few backward tadpoles, the suckers have disappeared, and it would seem that they now hold by their mouths when attached. They grow well and are very lively; but not more than three or four at a time are to be seen eating the confervæ, and they not show any remarkable eagerness for food.

32 (2). A drawing was made of the upper surface of a tadpole seen by transmitted light. The head and body were bordered by a transparent fold of the skin. The opening out of the gill-chamber on the left side was very plainly seen. The vent was open, a quantity of fæcal

matter having been lately voided. The rudiments of the hind limbs continued small. The back was beautifully spangled with gold.

Plate X.

33. A drawing was made of the ventral surface of a tadpole. It agreed in every respect with that made on the 31st day.

34. A living tadpole was examined under the microscope with a two-thirds object-glass. The water contained great numbers of monads; and by watching them, it was seen that a strong steady current set in through each nostril, and a corresponding stream passed out through the opening of the gill-chambers at the left side of the body.

A drawing was made of the outlet from the gill-chambers. The nostrils had the form of short wide projecting tubes.

34 (2). The gold spangles were perfectly metallic in appearance; and their angular irregular shapes exactly imitated small bits of gold leaf. Sketches of three of them were made.

35. A drawing was made of the ventral surface of a tadpole, inclining to the right side and viewed by transmitted light. It showed nothing new, but gave good representations of the mouth and the rudiment of the right hind limb.

36. A tadpole was killed, and the gold spangles on the back examined. They were much smaller than before, and more numerous: almost all of them were in pairs; but occasionally there were three in a row. Sketches were made of them.

37th. On the back of a tadpole examined today the gold spangles were again smaller than before, and were in groups of three or four instead of in pairs. The rudiments

of hind legs remained small; they were translucent and showed no trace of internal structure.

38. Drawings were made of two tadpoles, the natural size.

During the last week the tadpoles have eaten large quantities of confervæ, fresh supplies of which have been given to them daily.

Plate XI.

48th. Before leaving home for ten days, the tadpoles in the large basin were supplied with a quantity of confervæ scraped from the sides of the pond; on my return the whole of it was found to be consumed, and the quantity appeared to have been insufficient.

48. The tadpoles in the pond were double the size of the largest in the basin; and several of the latter were comparatively very small. Drawings, of the natural size, were made of one from the pond and of a large and a small one from the basin, showing the difference there may be in size in tadpoles of the same age. The development of the tadpoles in the pond is no further advanced than that of the others.

48 (2). A drawing was made of the mouth of one of the larger tadpoles from the basin, under the microscope with the one-inch object-glass. It showed the brown horny beaks, the upper being the larger and overlapping the lower; both had their cutting-edges serrated. The upper lip had two rows of comb-like teeth, and the lower had four rows of similar teeth; the outer row, however, was very small. The lips were white, and were foliated and frilled, and very large compared with the opening of the mouth.

48 (3). A drawing was made of the angle between the body and the root of the tail on the right side; it showed

the rudiment of the hind limb in the same condition as when last represented (on the 35th day).

The whole of the upper surface of the body of this tadpole was covered with gold spangles; they were small but very numerous, and were often arranged in groups of four, forming a square; and sometimes four or five were near together in a line. The spangles on the under surface of the body, which were fewer in number, were like silver. The muscular middle part of the tail was thickly covered with stellate marks of black pigment; a few were also scattered over other parts.

49. One of the largest tadpoles from the basin was killed, and a drawing made of its mouth with a two-thirds object-glass under the microscope. It was placed rather sideways; and the lower lip was not everted; so that, although the whole of the details were not fully shown, it gave a natural view. The lips and the teeth upon them are fitted for holding food, and the strong serrated beaks for biting.

51. A tadpole from the pond was examined living. Water was seen to flow in through the nostrils, and out from the opening at the left side of the body. The gold spangles were small and distributed in clusters of perhaps a score in a patch.

A drawing was made of the right hind limb (51). Development had commenced; the limb was short and thick, contracted slightly at the ankle and club-shaped at the end; its shape was something like that of a horse's foot.

53. An outline drawing of a large tadpole from the pond was made, of the natural size.

Plate XII.

Two drawings were made, one of the left, 53 (2), the other of the right, 53 (3), hind limb of a tadpole from the pond. They showed an advance from the former state,

the difference being that they were considerably larger, and the club-shaped termination was broader and flatter, and was marked by a faint indication of division into toes.

53 (4). By removing the skin of the ventral surface of the body, the viscera were exposed: the parts seen were the heart, right gill, liver, intestine, and pancreas. The liver consisted of only two lobes, which were comparatively small; and in the course of that part of the intestine which was exposed to view was a globular enlargement apparently in connexion with the pancreas.

54. A drawing of the left hind limb was made; it showed the divisions of the toes slightly more distinct than before.

55. A dozen tadpoles were taken from the pond; and a drawing was made of the largest, the natural size. The tadpoles were turned into a large basin of water and watched; they were seen to come to the surface from time to time and put out their mouths, from which a small bubble of air escaped.

55 (2). The tadpole of which an outline drawing is given was killed. The hind legs were considerably advanced, the toes being partially divided, and the knee and ankle-joints indicated by bendings.

Plate XIII.

55 (3). On taking off the skin from the chest and abdomen, the parts brought into view were the heart in the middle and the gills on each side; a transverse partition separated them from the contents of the abdomen, which, so far as they were visible with very little disturbance, were the liver on the right side and the intestine on the left, the coils of which were slightly opened out so that they could be better seen. An enlarged drawing was made of the whole body.

55 (4). A drawing was made of the right gill as seen under the microscope with the two-inch object-glass.

56. A score of tadpoles were taken from among the dead leaves in the pond at one sweep of the dredge. The largest was selected for examination. A very strong constant current came out through the outlet of the gill-chambers; but no current was seen entering the nostrils. The mouth was constantly opened and closed with an action like swallowing.

The tadpoles were rapidly increasing in size. Dense crowds of them congregated in the morning at the shallow part of the pond, where the bottom was covered with a thick layer of dead leaves; they frequently came up to the surface for a moment, as newts do to breathe. In the afternoon they presented a curious sight: they had left the dead leaves and were spread all round the sides of the pond, grazing on the confervæ.

57. A tadpole was killed and a drawing made of the right hind leg. The toes were distinct; the first, second, and third were deeply separated from each other; the two outer ones were less formed. The knee-joint was indicated by an angular bend.

58. A drawing was made of a tadpole, the natural size, representing the right side.

58 (2). The right hind leg was then drawn, under the microscope with the two-inch power. It was well formed, showing the thigh, knee, leg, and foot, the five toes of which were distinct though short and stumpy.

In the morning the tadpoles were, as usual at that time, congregated amongst the dead leaves; but in the afternoon they were distributed round the borders of the pond, feeding eagerly on the confervæ, a great part of which had disappeared.

Plate XIV.

59. A drawing was made of the mouth of a tadpole. The teeth on the lips were in the form of combs—that is, in long rows set in narrow white bars which appeared to consist of cartilage. In the upper lip were five sets, the first or outermost of which was complete from side to side; the second formed a pair not meeting in the middle; the third consisted of a short length on each side, the fourth a longer set on each side, and the fifth a very small comb on each side with only three small teeth. The lower lip had four sets, all of them divided at or near the middle.

59 (2). A drawing was made of the outlet from the gill-chambers, as seen from above.

59 (3). On laying bare the right gill the toes of the right fore limb were found pointing forwards; the limb was traced backwards, and a drawing made of as much of it as could be exposed. Its development was about equal to that of the hind limb of the same tadpole.

60. The skin of the right side of a tadpole was cut away, exposing the right gill with the right fore leg resting upon it, the liver, and a loop of the intestine. An enlarged drawing of the whole animal was made showing these parts.

60 (2). A drawing was made of the right fore leg under the microscope with the two-inch object-glass.

60 (3). A corresponding drawing was made of the right hind limb, showing the comparative size and the agreement in development of the two.

The food of the tadpoles has been exclusively vegetable; they are seen constantly grazing on the confervæ round the sides of the pond.

Plate XV.

61. A drawing of the ventral surface of a tadpole was made, of the natural size.

61 (2). It was next turned on its right side and a slightly enlarged drawing of the left side was made, showing the opening into the gill-chambers.

61 (3). The skin was then cut away from the abdomen, and a drawing was made, natural size, showing a front view of the helix-like coil of intestine undisturbed. This view represents only a small part of the intestine, the coils of which are several layers deep.

The tadpoles in the pond, which were unusually large, were a wonderful sight; they were in dense crowds feeding on the confervæ.

62. A slightly enlarged drawing of the left side of a tadpole, in which the gill, fore limb, and coils of intestine were exposed, was made.

62 (2). The intestine was then pulled out and a drawing made of the animal, natural size, with the intestine partly uncoiled.

63. A drawing was made of the right fore limb and gill of a tadpole under the microscope with a two-inch object-glass. The gill had an unusual appearance, a considerable part of it consisting of a smooth white substance.

Plate XVI.

64. Two outline drawings of a tadpole, natural size, were made—one a dorsal view, the other the left side; both showed the opening of the gill-chambers.

66. A drawing was made of the left fore limb, left gill, and heart of a tadpole under the microscope with the two-inch object-glass.

The tadpoles have now eaten all the confervæ from the

sides of the pond, and have retired to the bottom; none are anywhere to be seen.

68. The tadpoles having returned to the dead leaves at the shallow part, a number of them were taken out; and two of them were selected, which differed most in size and in the development of the hind legs. Outline drawings, natural size, were made of them.

68 (2). The intestine of the smaller specimen was divided close to the vent and drawn out to show its length, which was about three and a half times the length of the body. A drawing, natural size, was made of it.

69. An enlarged drawing was made of the left side of a tadpole, showing the opening from the gill-chamber. A dull-white band in the skin was seen to pass obliquely from the opening forwards and downwards under the front of the chest, marking the passage from the gill-chamber to the external opening, which was at the end of a short, conical, projecting tube, as represented in this and in all the other drawings of this part.

Plate XVII.

69 (2). A drawing was made of the mouth of the tadpole of which the enlarged outline is given on Plate XVI. 69. There were on the upper lip four lines of small brown teeth set in frames like combs: the outermost extended from side to side; the others were in pairs, each shorter than the one before it, the inner border of the lip, where it formed the opening of the mouth, being of an arched shape. On the lower lip were four lines of combs, the teeth of which were longer than those of the upper lip: the outermost consisted of two parts nearly meeting in the middle; the next and next but one were complete from side to side; and the innermost was formed of two short combs not meeting the middle.

70. A tadpole taken from the pond had a piece of intestine about half an inch long hanging from a small perforation in the wall of the abdomen near the vent. On cutting away the skin from the ventral surface, the whole of the great coil of intestine was found to be absent, none of the alimentary canal remaining except the short length which hung externally and a loop in close connexion with the liver. The left lung was seen filled with air; and its free end floated up in the water in which the tadpole was examined. When the liver was removed the right lung was also brought into view. An enlarged drawing was made, showing the heart, gills, and lungs.

72. A fine tadpole with the hind legs well developed and carried in a frog-like manner was taken; and a drawing was made, of the natural size. On examining the abdominal viscera the intestine was found almost empty.

72 (2). By removing the liver and intestine the lungs were exposed. They were of equal size, long, narrow, and tapering to a blunt point. A drawing was made of one of them under the microscope with the two-inch object-glass. It consisted of a sacculated tube composed of structureless membrane in which small pigment-cells were scattered.

Some pieces of meat were put into the pond to supply the tadpoles with food.

73. Crowds of tadpoles were eagerly feeding on the meat. Two were selected, one in an advanced stage of development, the other more backward. Enlarged drawings of the left side of each were made, showing the opening of the gill-chamber at the end of a short, projecting, conical tube, and also the remarkable change in the shape of the body as development advances.

73 (2). A drawing was made of the lungs of the larger of the two tadpoles.

Plate XVIII.

73 (3). An enlarged view of the larger of the two tadpoles (Plate XVII. 73), showing the liver and coiled intestine, was made. On the removal of these viscera, the lungs, as represented on Plate XVII. 73 (2), were exposed and the drawing of them made.

74. The tadpoles continued to feed eagerly on the meat. Among those taken this morning one had four legs; the shape of its body was frog-like, which seemed to depend on development of the skeleton, as seen in the form of the head and back. Another tadpole was taken in which the fore legs remained under the skin, but were on the point of being pushed out. Drawings of both were made, the natural size.

74 (2). An enlarged drawing was made of the ventral surface of the four-legged tadpole. It showed that the left fore leg had passed through the opening of the gill-chambers, whilst on the right side there was a distinct, well-defined, round hole, through which the right fore leg had come out. This, however, is not shown in the view given.

74 (3). On removing the skin from the ventral surface the gills were exposed, and were unusually distinct. A broad band of cartilage extended across the chest immediately behind them, and united the fore legs. The abdomen was small, and tightly bound in by a purplish sheet of membrane which was marked by a darker line down the middle from the centre of the shoulder-girdle to the pubes. The enlarged drawing represents these parts.

74 (4). By removing the purple fascia the liver and intestine were exposed to view, and appeared as represented in the drawing, the coil of the intestine being wonderfully diminished in size.

Plate XIX.

73 (4). An enlarged drawing was made of the liver and intestine removed from the tadpole represented on Plate XVIII. 73 (3). The intestine was large, long, and uniform in diameter throughout.

74 (5). A frog-like tadpole was examined. The gill-opening had its usual shape and size, not yet altered by the left fore limb pushing against it. A drawing was made of its mouth under the microscope with the two-inch object-glass. The lips were smaller than they are shown in previous drawings. The upper lip, from its position, showed only the one outer long row of teeth, with another row behind it on one side. The lower lip, being everted, showed all the teeth upon it clearly: the rows were much degenerated; one had disappeared entirely, and the other three were reduced to imperfect rows on each side. The teeth were irregular in length, and most of them small and as if falling out. The brown part of the beaks was only half the width of previous specimens, the serrations on their cutting-edges, however, were strongly marked.

75. A less-advanced tadpole was taken, and a drawing made of its mouth. It showed the lips and teeth within, fully developed and of the characteristic tadpole-form.

77. A tadpole in which the body had assumed the frog-form was examined. The gill-opening at the left side was large; and a toe of the left fore limb protruded through it. There was no appearance in the skin of the right side of a perforation, though it was baggy and thin over the right limb. An enlarged drawing of the ventral surface was made.

77 (2). On cutting away the skin from the ventral surface, the left fore limb was found with the palm turned outwards, and it was seen that the inner toe had pressed

through the gill-opening. The right fore limb had the palm turned as usual towards the body. The liver was larger than before, and occupied more than half the abdomen. The intestine was much shrunk; and the helix-like coil did not face forwards as formerly, but was at the left side and very much reduced in size. A drawing was made showing these particulars.

77 (3). The liver and intestine were removed; and an enlarged drawing of them was made with the intestine pulled out, to compare with that represented in 73 (4). The intestine was much shorter than it had been; and the part corresponding with the middle of the spiral was remarkably shrunk both in length and diameter.

Plate XX.

79. A tadpole in which the fore legs had lately come out was taken for examination. An enlarged drawing was made of the right side, showing that the right fore limb had been pushed out through a ragged hole in the skin.

79 (2). The left fore limb had come out through the gill-opening, which, being too small to allow it to pass, had its posterior border torn. This appears to be invariably the case. A sketch was made representing the part.

79 (3). The abdomen was wonderfully contracted; and the fascia beneath the skin was very distinct from it. On opening the abdomen, the greater part of the cavity was seen to be occupied by the liver. The gall-bladder was large, round, and of a bright-green colour. The intestine was quite empty and of a delicate flesh-colour, and it was reduced to a very small size compared with that of the tadpole in its earlier stages. The mouth was a transverse opening, with the lips consisting of the ordinary skin. An enlarged drawing was made of the liver and alimentary

canal; a slight dilatation at the upper part showed the incipient stomach.

83rd day. Yesterday a young frog was taken from the pond with a very small remnant of the tail. Today, on looking round the borders of the pond, about half a dozen young frogs were seen standing against the side with their noses just out of the water. Their sight was very quick; and on approaching them, they instantly swam away. Two were taken out with the dredge, and placed in a basin of water, where in a few minutes they became exhausted and would have died drowned if not taken out. They were placed on a saucer with a few drops of water, and were covered with an inverted tumbler, up the inside of which they immediately climbed and remained holding on to the glass. They had each a small brown stump of tail.

The young frog taken yesterday, on which no vestige of tail now remained, was examined. A drawing (83) was first made of it, the natural size. The shape of the body was strangely altered from that of the tadpole, the abdomen being drawn in so that the animal appeared on the verge of starvation.

83 (2). On removing the skin from the ventral surface, that over the abdomen was found to be loose and pale-coloured; but the fascia beneath was purple and very tightly bound down. The liver and alimentary canal were removed in a mass and an enlarged drawing made of them. The stomach was large, and widest at the cardiac end; it was placed across the abdomen close below the liver. The intestine was very short, and made two or three turns; it was quite empty. The gall-bladder was large, and filled with dark green bile.

83 (3). The loops of the intestine were opened out, and a second drawing made to show its whole length.

86. A frog with remains of a tail about three quarters of an inch long, was taken from the pond and examined. A drawing, natural size, was made.

86 (2). The skin was then taken from the abdomen; and after removing the dark-coloured layer beneath it, the viscera were exposed. An enlarged drawing was made of the parts seen; they were the liver, gall-bladder, stomach and intestine, and the left lung, which chanced to be floated up out of its natural position. The gall-bladder was remarkably large, and, as usual, globular in shape and of a dark bottle-green colour.

86 (3). An enlarged drawing was made of the liver, stomach, and intestine, removed from the body. The intestine was quite empty.

86 (4). The left lung was cut off and a drawing made of it under the microscope with the two-inch object-glass. It was floated on water and viewed by reflected light.

Life-history of the Tadpole.

The foregoing observations, though superficial and incomplete, appear to afford sufficient material for an outline sketch of the life of the tadpole, which may be filled in by the results of more elaborate researches. The development of the different organs in orderly succession suggests the division of the life-history into periods, each distinguished by the appearance of certain organs or structures; and five such periods are naturally marked out—the first extending from the deposition of the ovum to the escape of the young tadpole from the egg, the second from this time to the completion of the alimentary canal, the third to the commencement of the development of the limbs, the fourth to the commencement of those changes by which the larval form of alimentary apparatus

is converted into that of the frog, and the fifth to the complete absorption of the tail.

First Period.—This includes the segmentation of the yelk and the development of the embryo. According to the present observations, the process of segmentation occupied eight days, and seemed especially to linger during the sixth and seventh days, when no change could be perceived in the ovum. In the drawings representing the successive stages of the formation of the embryo the omission of the first stage is unfortunate, as it makes the change abrupt from the mere spherical egg to a form with bilateral symmetry, in which every portion of the yelk-mass has taken a definite position in relation to the median groove. Allowing for this omission, the series of drawings of the embryo within the egg-membrane gives a fair representation of the progressive advance in form and proportion until the tadpole is ready to escape.

The rudimentary structures which appear during this period may be stated to be :—the cerebro-spinal axis and its supporting skeleton, with the continuation of these structures in the large powerful tail; the enclosing body-wall; the visceral arches; the organs of sense; the heart, and the blood-vessels of the developing parts, which include the middle dorsal line and fore part of the body, namely the head and throat; and the general cutaneous system. But the cavity of the abdomen remains filled with undifferentiated yelk-mass; and no commencement of the organs which it afterwards contains is at present made. The branchial arches are considerably advanced before the close of the period; and in this condition as regards the structures which are permanent throughout the life of the tadpole the animal is hatched.

Two other organs, however, have been formed, the use of which is temporary and confined to the second period;

these are the sucker and the external gills; and they are already so far developed as to be perfectly efficient when the tadpole escapes from the egg-membrane. The sucker is at first large and single, extending across the middle line; and it is in this condition when the animal is hatched. The external gills, two on each side, have budded out from the skin over the middle of the first and second branchial arches, and have, before hatching, attained such a size as to serve at once as organs of respiration. There is no regularity in their form or in the number and arrangement of their divisions; most frequently they consist of an irregular bunch of filaments. The time occupied by this period was, in the present case, fifteen days.

Second Period.—The special temporary organs in use during this period are the sucker and the external gills. The developments which take place in it are those of the branchial arches and clefts, the internal gills, the eyes, the gill-chambers, formed by the opercular fold, which commences to grow on the second day and is completed on the fourteenth; the passages of the nostrils to the back of the mouth, these being in connexion with the internal gills, the ciliary action of which draws the water through them; and the suctorial beaked mouth and coiled intestine peculiar to the tadpole. At the close of the period very small rudiments of hind limbs appear.

This second period is a continuation of the embryonic condition; contact with water and aquatic respiration are necessary for further development; but rest is also required, and the animal continues as stationary as whilst in the egg. It is blind and helpless and cannot feed, having no alimentary canal. Immediately on its escape the tail is used, but only to swim to some fixed object, to which it attaches itself by its sucker; and there it remains until the close of the period, unless forcibly detached, when it at once refixes itself.

In the course of the first few days the single sucker divides into two; and though these continue efficient almost to the end of the period, they daily diminish in size, and have disappeared at its close. I might perhaps have concluded that the single sucker was an abnormal condition in the individual selected, if two other tadpoles examined on successive days had not shown remains of the same peculiarity.

The external gills, efficient at birth, attain their full size on the second day, after which they gradually shrink, but continue in action for about a week, soon after which what remains of them disappears within the opercular fold. The developed state of the external gills before hatching was common to all the specimens under immediate observation, which were at least a hundred in number; and I have noticed the same fact on other occasions.

The clefts between the branchial arches, formed very soon after hatching, have no relation to the external gills, which hang freely in the water and extend, speaking comparatively, to a considerable distance from the body; but the large cilia on the surface of the lower side of the head change the water in which they are bathed by the strong current they produce.

The organs which are developed during this period are all completed at about the same time—that is, very nearly at its close. The eyes and nostrils progress regularly from the time of hatching; but with regard to the internal gills and the alimentary canal, it appears that the former are commenced first, and are probably in use about the fifth or sixth day in conjunction with the external gills, gradually increasing in efficiency as the latter diminish in size; but the complete apparatus for the tadpole-respiration is not perfected until the opercular fold has joined the skin of the abdomen, leaving only a small opening at the

left side, and until, in conjunction with the chambers thus formed, the nasal passages are opened into the back of the mouth. This use of the nasal passages for the aquatic respiration is not mentioned by any of the authors I have consulted, and appears an observation of some importance; for the water so entering leaves the mouth entirely free for feeding.

The development of the peculiar tadpole-mouth and the enlargement of the abdomen, indicating the formation of the coiled intestine, commences only when the gill-chambers are complete; and these structures are made ready for use within the last three days of the period. It is during these three days also that the remarkable gold spangles in the skin are seen in their greatest perfection. Whilst these developments have been taking place the tadpole has grown much larger, and the tail especially has become more expanded and stronger, so as to form a powerful aquatic locomotive organ. This second period, like the first, occupies fifteen days.

Third Period.—The third period, extending from the thirtieth to the fiftieth day, appears to be devoted entirely to growth, no new developments being observed. The tadpole swims actively, feeds voraciously, and grows very rapidly. The gills are contained in a chamber formed by the opercular membrane; and water, entering by the nostrils, passes through the branchial clefts and, after bathing the gills, escapes through the opening at the left side. This opening, which is at first formed by the mere absence of union between the free edge of the opercular fold and the skin of the abdomen, and is seen near the left border of the ventral surface, soon becomes tubular, and gradually extends upwards and backwards, embedded in the skin to about the middle of the left side of the body, where it terminates as a short conical projecting tube.

The mouth consists of a pair of strong brown horny beaks with serrated edges, and is surrounded by a broad expanded sheet of white fleshy membrane, quite distinct in character from the skin of the body; it is irregularly frilled and scolloped at the edges, and, being continuous at the sides, forms, when spread out, an irregular circular figure, in which the distinction of upper and lower lip is marked only by position. The inner or front surface is furnished with long ranges of small brown teeth set upon white frames or backs, so that they resemble combs. There is some irregularity in their number and arrangement in different specimens; but usually there are three sets in the upper and four in the lower lip. The two inner sets in the upper lip are interrupted in the middle by the arched form of the upper beak; and there are also often one or two pairs of very small combs, each with only three or four teeth, placed between the longer sets. The food of tadpoles is indifferently vegetable or animal, but more commonly vegetable; when animal it consists of the flesh of any dead creature that may chance to be in the water. The lips are applied in the manner of a sucker, and give a firm hold for the action of the beaks.

The alimentary canal consists of a long simple tube of equal diameter throughout; and from the point where it leaves the liver to its termination it is laid together double, and is coiled round and round, filling the left half of the abdominal cavity, and forming in its final coils the regular flat spiral, with the loop of the doubled intestine in the centre, which is seen on removing the skin of the ventral surface of the abdomen. This period occupies twenty days.

Fourth Period.—In this period, extending from the fiftieth to the seventy-fourth day, eager feeding and rapid growth continue. The developments which take place in

it are those of the limbs, with the shoulder-girdle and pelvis, and the lungs. The gills continue active; water, however, no longer enters through the nostrils, but through the mouth, and the nostrils have become very much smaller in proportion to the increased size of the body.

Small rudiments of the hind limbs in the form of a pair of papillæ were visible at the close of the second period; and they continued without change through the third period, except that they increased in size with the increased size of the body; but at the commencement of the present period (that is, on the fiftieth day) the ends of the papillæ enlarge and become club-shaped. Afterwards they grow broader and flattened, and the divisions of the toes begin to appear: these are well-marked on the fifth day, when the knee and ankle are also indicated by bendings in the lengthening limbs. In ten days more both fore and hind limbs may be said to be fully formed, though the toes of the hind pair are still without webs. The fore limbs, budding out at the back of the gills within the gill-chamber, and also being much smaller than the hind limbs, were not seen in their earliest stages; but when examined on the tenth day they were found to agree in development with the corresponding hind limbs. During the remaining fourteen days both fore and hind limbs grow rapidly, and the hind toes become webbed; at the close of the period the shoulder-girdle and pelvis are completely formed.

The development of the lungs appears to proceed gradually during almost the whole of the fourth period. The first indication of their presence was on the fifth day, when the tadpoles were seen to come to the surface from time to time and put out their mouths, from which a small bubble of air escaped. These organs, however, were not actually examined until the twentieth day, when they

had attained a considerable size, and had each the form of a sacculated tube, bluntly pointed at their free end. During this period the liver increases much in size, and additional lobes are formed.

Fifth Period.—The last period of the aquatic life of the frog extends, according to the present observations, from the seventy-fourth to the eighty-third day, at which time the animal has left the water and attained the perfect form of the adult. At the commencement of this period it is a tadpole, with a large powerful tail, a pair of large hind legs with webbed feet, and a pair of equally developed fore legs still retained beneath the opercular fold which forms the gill-chamber. The gills are still in action; and the opening at the left side remains for the discharge of water.

Now, however, the animal ceases entirely to feed, and the alimentary canal soon becomes empty; the large white frilled lips shrink, and the sets of small teeth upon them degenerate and drop off; the horny beaks disappear; the mouth widens; the new lips consist of ordinary skin; and the large fleshy bifurcated tongue of the frog is formed. At the same time the intestine shrinks both in length and diameter, especially at the part which forms the centre of the coil; soon afterwards its upper part begins to be dilated to form the stomach; and the remainder becomes rapidly smaller and shorter as the stomach enlarges, until it consists of a small and slender gut, making only two or three short turns below the stomach at the left side. The liver at the same time enlarges; and the gall-bladder is seen, very large and filled with dark green bile.

When these changes in the alimentary apparatus have been partly accomplished, the fore legs are pushed out, the action being instantaneous for each limb. Sometimes the right, sometimes the left comes out first; and the left

limb is always forced through the gill-opening, the margin of which is torn in consequence of the size of the natural aperture being too small to allow its passage. On the right side the opening is usually a ragged rent in the skin, which has become thin at that part. The effect of the protrusion of the fore legs is at once to stop aquatic respiration and throw all the work of breathing on the lungs: the animal remains almost constantly on the surface; and it very soon becomes exhausted and drowns if it cannot support its nostrils above water.

The tail now rapidly shrinks, being absorbed and used as stored-up nutriment by the animal during the time when it is unable to feed. It remains several days in the water in this state; but when the tail has been reduced to a brown stump about half an inch long, and the alimentary apparatus is completed and fit for its approaching new mode of life, the animal shows an instinctive desire to climb. Reaching the border of the pond, it will make its way up a steep or even perpendicular surface to the height of several feet, if this be necessary in order to reach the bank. Here, among the grass, numbers of them may be seen leaping about, still with a stump of tail, which serves as provision for several days longer, after which, when it has entirely disappeared, the young frog has gained sufficient experience to take small insects as food.

In this life-history of the tadpole of the common frog, the dates, given with the observations and drawings, have been made use of to mark the number of days included in each period; but no special value is intended to be set upon the exact length assigned to each; for, in strictness, the dates refer only to the particular tadpoles examined. They are true, however, for this one instance; and they may be of use for rendering more clear the proper order of succession of the different developments. Tadpoles hatched

from the same mass of frog-spawn differ considerably at all stages of their growth, both in size and degree of development; and this difference is seen from the commencement. The ova on the surface of the mass develop, and the tadpoles escape, whilst those in the centre still remain unchanged, at least as regards their round shape. The hatching of the ova, in fact, instead of being simultaneous, takes place successively from the surface to the centre; so that if the earliest tadpoles escape in about a fortnight, the latest will not follow until a week or more afterwards, the cause apparently being that they require direct contact with water for their development. And this fact is of interest, as it may be some advantage to the tadpoles by allowing them gradually to disperse, and thus preventing the excessive crowding which would result from such immense numbers being hatched at once at the same spot. This difference in the time of hatching accounts to some extent for the difference in size and development seen in the tadpoles afterwards; but a further cause of difference appears to have reference to feeding, the greater natural vigour of some enabling them to take a much larger quantity of food, whilst weakness of constitution, or, perhaps, some defect of the alimentary apparatus, throws others far behind. But, at the same time, the great majority of the tadpoles do not differ very widely in development, about a week or ten days representing the amount of their variation.

For the purpose of bringing the successive periods as nearly as possible to one standard, the largest and most advanced tadpole was always selected from a considerable number taken out of the pond for each daily observation, the others being put back; and in consequence of this method all the periods as given are rather shorter than the averages would be. The extremes will be sufficiently

shown by the fact that the first young frog left the water on the eighty-third day, and the last not until the one hundred and third, or twenty days later. The influence of temperature in hastening or retarding development has no bearing on the present observations, the ordinary spring temperature in which the ova were hatched presenting no extremes which could produce perceptible effects.

XI. *On the Levenshulme Limestone: a Section from Slade Lane eastwards.* By Wm. Brockbank, F.G.S.

Read March 6th, 1883.

The Permian strata around Manchester were favourite subjects of study with our former President, the late E. W. Binney, and his communications on them were frequently laid before the Society and printed in its 'Memoirs.' To Mr. Binney also were the Surveyors who mapped the geology of the district for the Government indebted for most of the information they used in laying down the Geological-Survey plans of the Permian and Coal-measures. This is acknowledged by Mr. E. Hull in his descriptions of the geology of the country about Oldham, Bolton Le Moors, and of the Permians of the Midland Counties. When, therefore, the Levenshulme limestones at Slade Lane first came under my notice I at once referred to Mr. Binney's papers, and found a section laid down about two

miles to the south of it, from Heaton Mersey to Goyt Hall, beyond Stockport, which gave me the clue to the solution of the new problem; and I have followed Mr. Binney's views in the following communication.

In making a sewer from Withington to Levenshulme, the strata underlying the covering of Lower Boulder-clay have been brought to light for about two miles, commencing with the New Red Sandstone and passing through the Permian series of red clays, shales, and sandstones until the Levenshulme limestone was reached. I do not propose to enter much further into this part of the subject, as Prof. Boyd Dawkins is engaged upon it. There is, however, one point which falls within my province, as I have taken pains to investigate it; and this is, that from Withington eastwards the strata appear to have been found continuously, without any fault to disturb the sequence, in the passage from the Triassic to the Permians, and down to the Levenshulme limestone at Slade Lane. This is roughly shown by a section prepared as the works progressed by Mr. Swarbrick, the Surveyor to the Withington Local Board, to whom I am indebted for this and other valuable information. The section now produced shows red rocks and measures all the way, the surface varying very considerably and being covered with from ten to twenty feet of Lower Boulder-clay. The more immediate subject of my paper is based upon the observations made by Mr. Harper, the Surveyor of the Levenshulme Local Board, who has kindly furnished me with a copy of his working-plan and section, together with answers to many queries I had submitted to him for this object.

A sewer has been made under Mr. Harper's superintendence, commencing in Levenshulme at the Manchester and Stockport road, passing westwards under the L. & N. W. Railway at Levenshulme Station, along Albert Road

to Slade Lane; it then turned southwards until a junction was made in Slade Lane with the sewer above described (from Withington to Slade Lane) at Cringle Brook.

These works commenced at Levenshulme in the Lower Boulder-clay and continued in the same all along the Albert Road westwards to Slade Lane, near which some very large lumps of limestone were met with, from a ton to 30 cwt. each. On turning southwards along Slade Lane the red clay was entered opposite the Grange, and was drifted through for about 270 feet. It came within about 12 feet of the surface, and was covered with the drift-clay. The Levenshulme limestone was next reached and was found to be lying upon the red clay. It was drifted through altogether for about 740 feet. The first limestones were about two feet thick and bedded, with red-shale partings. The limestone itself is of a delicate grey pink, and when polished takes a marble lustre. It came at one point within nine feet of the surface, but had a very irregular outline and was much worn. The next was through solid limestone, the course of the workings being nearly in the line of the face of the rock. Upon this limestone lay a band of ironstone six inches thick. The drift then passed through broken metals for about sixty feet, after which solid limestone was again entered and passed through for seventy feet. The strata now became much disturbed by a fault, as marked on Mr. Harper's section. Solid limestone was again found beyond the fault, followed by broken metals of brownish rock. The red sandstone was entered at Cringle Brook where the Withington sewerage works joined, and, as has already been shown, the red measures continued thence to Withington, passing from Permians into the Trias. The above description gives the impression that there were several seams of limestone, but I do not think this was the case. The drift appears to have run

along the face of the outcrop or near it; and the strata being much disturbed, it is impossible to say the exact thicknesses of limestone seams or if there were more than two or three. It would be very desirable to put down a bore-hole or to sink a shaft at a suitable point, so as to prove exactly the stratification at this place. Any detail-sections, such as those now submitted to the Society, must be taken as merely diagrams built upon the evidence above stated, and not as actual records of ascertained facts.

When the Levenshulme limestones were first discovered specimens of them were sent to Owens College, and Professor Boyd Dawkins at once pronounced them to be Ardwick limestone, and belonging to the Upper Coal-measures, chiefly because they contained the *Spirorbis carbonarius*.

For my own part, I prefer to call them the Levenshulme limestones, and to class them as Permian, following Mr. Binney in his description of what I take to be a similar limestone, shown in his Heaton Mersey section, about two miles to the southwards, and thrown up by the same fault. I am aware that, judging by the fossils named by Mr. Dawkins, the limestone may be classed as a member of the Upper Coal-measures. But we have here at Withington a regular sequence of Permian red measures ending in the Levenshulme limestone, which is again underlain by red clays, as at Heaton Mersey; and I therefore consider the lithological evidence to be in favour of its being a Permian limestone. It is a subject quite open to discussion before we consent to differ from Mr. Binney's ruling; but I do not propose to discuss it further at this time.

A considerable quantity (twenty tons or more) of the Levenshulme limestone blocks have been removed to my garden, at Brockhurst, Didsbury, where they are perma-

nently placed so as to be accessible for examination. They present many points of great interest in addition to the fossils which they contain in plenty. Some large blocks, over a ton in weight, have weathered and worn surfaces, bearing evidences of having formed the outcrop for some time before they were covered up by the boulder-drift. Mr. Boyd Dawkins saw some of them *in situ* whilst the drift was in progress, and he informed me that the appearance was like that of the creviced top of a limestone hill in the Craven district; and some of these blocks have quite this appearance, being coated with carbonate of lime on several sides. There are some which appear to have formed a floor, having been planed down to an even surface, showing the veins of crystal very distinctly, as in a marble chimney-piece. And one large block of a ton weight bears unmistakable evidences of glacial action, having well-marked striæ and a polished surface. These particulars have been pointed out to several geologists who have inspected the blocks, and may be taken to be accepted facts. I think, therefore, we may safely say that the Levenshulme limestone formed an outcrop-ridge for some time subject to atmospheric influences before it was covered up by the Boulder-clay, that it bears evidence of glacial action, thus showing the agency by which it was altered to its present aspect and finally entombed.

Passing eastwards from Slade Lane, it has already been stated that the Boulder-clay was found to exist to a considerable depth, as shown by Mr. Harper's section. I have also ascertained from Mr. Worthington, the Engineer of the L. & N. W. Railway, that in the deep foundations for the bridges just erected in widening the railway at Levenshulme, no rock has been found, although it was sought for to a considerable depth. I have also made inquiry in the village, and cannot learn of any case from

well-sinkers and contractors where the clay has been penetrated. At the Levenshulme Print Works, half a mile further eastwards, an Artesian well was sunk some years ago by Messrs. Aitken Bros., and by the kindness of Mr. Thos. Aitken I now exhibit the section, the particulars of which are given in the Appendix. It shows the thickness of the Lower Boulder-clay and sands to have been 82 feet in all to the New Red Sandstone, and the total depth penetrated before the water-supply was obtained, 450 feet, the borings leaving off in the New Red Sandstone pebble-beds.

Half a mile further eastwards, at a place called Sandfold, some deep borings were taken, with a view to Sir Joseph Whitworth's proposal to purchase a site for his works; but the sands and clays were found so thick that it was abandoned. The Upper clays here come in, and a great thickness of sand intervenes, producing much water. It is interesting to note, by the way, that Mr. Aitken has in his possession a fragment of a Roman "cinerary"-urn found at Sandfold in a thin black seam which occurred in the sand-pit. It is thus clearly shown that from Slade Lane eastwards, for a considerable distance, the Boulder-clay is of immense thickness, some 80 feet in fact; and, supposing it absent, we should have a very marked ridge formed by the Levenshulme limestone, at least 70 feet above the Red Sandstone floor which lies before it. Whether this arose from denudation or from the effect of the fault is a problem which further researches may possibly solve. I incline at present to atttribute it to the upheaval of the strata to the west of the great fault at Slade Lane.

It will be seen that in this section we start from the New Red Sandstone at Withington and end with the same formation at Levenshulme, that the limestones at Slade Lane are brought to the surface by a great fault, and that

the Permian measures dip gradually to the west, passing regularly under the Trias at Withington.

By reference to the Geological-Survey map it will therefore be seen that there is a most serious error in the plans. The deep orange-colouring showing the Permians to the east of the railway at Levenshulme must be altogether erased and the New Red Sandstone substituted. Then, again, the Permians must be shown commencing at Slade Lane and extending for a considerable distance towards Withington, where the New Red Sandstone is at present shown. The line of fault from Heaton Norris to Collyhurst must also be placed much further west, on the line of Slade Lane. This fault is shown on the Geological-Survey map at Heaton Norris as the prolongation of the great boundary fault of the Manchester coal-field, being described as a downthrow to the east of 200 yards. It will, however, be evident from our new section that at Levenshulme it is rather an upthrow-fault to the west, as the Permians are brought to the surface, the Trias forming the platform to the east, and also two miles to the west of it.

The same fault is shown in the geological section of Manchester as passing near All Souls Church, Ancoats, where the Coal-measures are brought in to the east of it, and the Permians lie in very nearly the position they occupy at Levenshulme to the west of it. The same fault is again shown at Collyhurst, where the Ardwick limestone comes against it. The Collyhurst section is taken in very disturbed and disputable ground, and cannot with any certainty be relied on.

It is, however, clear, from a careful consideration of the line of fault shown upon the Survey-maps from Heaton Norris to Collyhurst, as altered by the new discovery at Levenshulme, that the rest of its course will require carefully seeking out, as it is not easy to bring it in at either

end without further information*. The whole of the Withington district will require also careful revision, as there is a large area of Permian measures not shown at all on the maps.

XII. *On the Transformations of a Logical Proposition containing a single Relative Term.* By Joseph John Murphy. Communicated by the Rev. Robert Harley, F.R.S.

Read November 28th, 1882.

The present paper has been suggested by De Morgan's "On the Syllogism, No. IV., and on the Logic of Relations," from the 'Transactions of the Cambridge Philosophical Society,' vol. x. part ii.

I indicate absolute terms, that is to say the terms between which the relations subsist, by Roman letters, and the relative terms by Italic ones.

So long as the relation is not transitive, R shall be taken to indicate any relation, and R^{-1} the converse relation. The relation of teacher, and the converse relation of pupil, are not transitive; that is to say, if X is a teacher of Y

* Mr. Binney (Lit. & Phil. Trans. vol. ii. pp. 31, 32) says that the existence of this fault was first pointed out to him by Mr. Hull; and he adds, "On taking the direction of the fault at Heaton Norris, it appears to run more in a line through Chorlton-on-Medlock, into the great Pendleton fault in the valley of the Irwell, than that of Bradford and Clayton." It will be thus seen that Mr. Binney's observations point in the direction of the Slade Lane discovery; but I do not think he is correct in his further surmise. There is another fault at Heaton Mersey which appears to be a continuation of the great Pendleton fault.

and Y of Z, this does not, in any common or natural sense of the word, constitute X a teacher of Z.

The denial of these relations shall be indicated by r and r^{-1}. For our present purpose we shall use these symbols with the following significations:—

R	teacher.
R^{-1}	pupil.
r	not-teacher.
r^{-1}	not-pupil.

These terms are to be understood in a purely qualitative sense; that is to say, the equation

$$X = RY,$$

and its inverse

$$Y = R^{-1}X.$$

mean respectively that X is a teacher of Y, and that Y is a pupil of X, without implying whether or not X has any other pupils and Y any other teachers. As in Boole's notation, the coefficient 1 signifies all or every; and I also use the coefficient 1^{-1}, which is new—at least I have not seen it used before. If

$$X = 1RY$$

means that X is all the teacher of Y or, in ordinary language, that X is the only teacher of Y, its inverse

$$Y = 1^{-1}R^{-1}X$$

means that Y is the pupil of none but X, or of X only. Thus, when standing before a relative term, 1 has the meaning of *only* when used as an adjective, and 1^{-1} has the meaning of *only* when used as an adverb. The equation

$$1^{-1} = 1,$$

which is true in arithmetic, is not generally true here; but it is true when

$$R^{-1}=R.$$

If R, for instance, means class-fellow, the truth of one of the two equations

$$X=1RY \qquad \text{and} \qquad X=1^{-1}R^{-1}Y$$

will imply the truth of the other. That is to say, if X is the only class-fellow of Y, then X is the class-fellow of Y only, or of none but Y. Of course it is here understood that neither X nor Y belongs to more than one class.

When the absolute terms X and Y are not the names of individuals but of classes, as is usually the case in the common logic, we shall use as the copula the sign of inclusion ($<$) instead of the sign of equality ($=$). When only one of the two terms X and Y is quantified by the coefficient 1 or 1^{-1}, the proposition is singly total; when both are so quantified, the proposition is doubly total. Thus,

$$1X<RY$$

means that every X is a teacher of a Y; and

$$1X<R\,1Y$$

means that every X is a teacher of every Y.

The contrary of a singly total proposition is doubly total, and *vice versâ*. Thus the two following are contraries:—

$$1\,X<R\,Y; \qquad\qquad 1\,X<r\,1\,Y.$$

That is to say, "Every X is teacher of a Y (or Y's)" has for its contrary, "Every X is not-teacher of every Y," or, in commoner language, "No X is teacher of any Y."

A singly total proposition of the above form admits of the following four forms, whereof the truth or falsehood

of any one implies the truth or falsehood of the rest, x and y signifying respectively not-X and not-Y, as in De Morgan's notation :—

$$1\,X < R\,Y.$$

$$1\,r\,1\,Y < x.$$

$$R^{-1}\,1\,X < Y.$$

$$X\,r\,1\,Y < 0.$$

That is to say :—

Every X teaches a Y.
Whoever does not teach any Y is a not-X.
Pupils of every X are contained among Y's.
There is no X who does not teach some Y.

A doubly-total proposition of the above form admits of the following four pairs of forms :—

$1\,X < R\,1\,Y.$	$1\,Y < R^{-1}\,1\,X.$
$1\,r^{-1}X < y.$	$1\,r\,Y < x.$
$1\,X < 1^{-1}ry.$	$1\,Y < 1^{-1}r^{-1}x.$
$X\,r\,Y < 0.$	$Y\,r^{-1}X < 0.$

That is to say :—

Every X teaches every Y.	Every Y learns from every X.
Whoever is a not-pupil of any X is not a Y.	Whoever is a not-teacher of any Y is not an X.
Every X is a not-teacher of none but not-Y's.	Every Y is a not-pupil of none but not-X's.
There is no X who does not teach every Y.	There is no Y who does not learn from every X.

The following equations are necessarily true:—

$$R \,\mathrm{I}\, Y = \mathrm{I}^{-1} ry,$$

and

$$R^{-1} \,\mathrm{I}\, X = \mathrm{I}^{-1} r^{-1} x$$

That is to say:—

A teacher of every Y is a not-teacher of none but not-Y's;

and

A pupil of every X is a not-pupil of none but not-X's.

"All knowledge is relative;" that is to say, only relations can be the objects of knowledge;—and we may treat the common logic as a particular case of the system expounded here, the relative term in this case being interpreted to mean identity or coexistence. Let us use C (the initial letter of *coexistent*) as the relative term; then c will signify non-identical or non-coexistent. "All X is Y" thus becomes "Every X is identical with a Y," or "The attribute X is always coexistent with the attribute Y;" and the proposition will be written, in our notation,

$$\mathrm{I}\, X < C\, Y.$$

The contrary of this, as shown above, is

$$\mathrm{I}\, X < c \,\mathrm{I}\, Y.$$

That is to say, "Every X is non-identical with every Y;" or, in common language, "No X is identical with any Y." Such a proposition is invertible;—if no X is identical with any Y, then no Y is identical with any X;—not because it is negative, but because it is doubly total.

We now go on to the application of this notation to transitive relations.

The following theorems are taken from De Morgan's paper already mentioned, page 16. I use L as the symbol of transitive relation, L being his symbol for relation generally. It follows from the definition of transitiveness, that

$$LL=L, \qquad \text{and} \qquad L^{-1}L^{-1}=L^{-1}.$$

If, then, L means ancestor and its converse L^{-1} means descendant, these equations assert that the ancestor of an ancestor is an ancestor, and the descendant of a descendant is a descendant. I quote De Morgan's theorems in his own words, with the same in my own notation above them; non-ancestor and non-descendant are indicated by l and l^{-1} respectively:—

(1) $L < L\,1\,L^{-1}.$
$< 1^{-1}\,l\,l^{-1}.$
$< l^{-1}\,1\,l.$
$< 1^{-1}\,L^{-1}L.$

(2) $L^{-1} < 1^{-1}\,L\,L^{-1}.$
$< l\,1\,l^{-1}.$
$< 1^{-1}\,l^{-1}\,l.$
$< L^{-1}\,1\,L.$

(3) $l < l\,1\,L.$
$< 1^{-1}\,L\,l.$

(4) $l^{-1} < 1^{-1}\,L^{-1}\,l^{-1}.$
$< l^{-1}\,1\,L^{-1}.$

(5) $l > L^{-1}\,l.$
$> l\,L^{-1}.$

(6) $l^{-1} > L\,l^{-1}.$
$> l^{-1}\,L.$

(1) "An ancestor is always an ancestor of all descendants, a non-ancestor of none but non-descendants, a non-descendant of all non-ancestors, and a descendant of none but ancestors."

(2) "A descendant is always an ancestor of none but descendants, a non-ancestor of all non-descendants, a non-descendant of none but non-ancestors, and a descendant of all ancestors."

(3) "A non-ancestor is always a non-ancestor of all ancestors, and an ancestor of none but non-ancestors."

(4) "A non-descendant is a descendant of none but non-descendants, and a non-descendant of all descendants."

(5) "Among non-ancestors are contained all descendants of non-ancestors, and all non-ancestors of descendants."

(6) "Among non-descendants arecontained all ancestors of non-descendants, and all non-descendants of ancestors."

XIII. *List of the Phanerogams of Key West, South Florida, mostly observed there in March,* 1872. By J. COSMO MELVILL, M.A., F.L.S.

Read before the Microscopical and Natural-History Section, February 13, 1882.

THE flora of this small island is limited and, naturally, to a great extent, maritime. In the year 1875 I recorded, in the 'Journal of Botany,' a catalogue of the Algæ—it is by no means so rich, proportionately, in flowering plants or ferns.

Key West, which is entirely of coral formation, constitutes one of many small reefs or "keys" which extend from Biscayne Bay westward to the Tortugas. Of these islets, this, which measures some seven miles in length by one to one and a half in breadth, is the only one inhabited, and it is not likely that the others would present many botanical novelties, though but few of them have been explored. Boca Chica, the next island, is separated at low water by only a very narrow channel. It is densely

wooded, and the character of the vegetation is entirely the same.

The south-western portion of Key West is cultivated, and towards the south and south-east, between the town and the sea, there are numerous green lanes and shady natural avenues extending for a mile or two, which present a native flora of much beauty, though somewhat scanty in actual species.

The northern shores of the island are swampy and there are extensive mangrove flats. It is in this portion that the original "scrub"-forest still remains, and the north-eastern shores especially present a deserted appearance.

In the centre of the island are numerous salt-pans and a salt-marsh, which abound in some curious plants, mostly Asclepiads and Chenopodiaceæ.

Chapman, in his 'Flora of the Southern States,' gives some Key-West localities, mainly on the authority of the late Dr. Blodgett, and Grisebach ('Flora of the British West Indian Islands'), in his ample table of distribution of each species, occasionally does the same. Mr. W. T. Féay, of Savannah, Georgia, lived for a year or more on the island and collected several species I did not observe. These have been added to this list, on his authority, to make it more complete.

In the Shuttleworth Herbarium, now incorporated with that of the British Museum, there are several specimens of Key-West plants, mostly collected by Rügel.

Owing to Key West being a town of increasing importance, as it is a military as well as a naval station, and also a calling point for all the steamers plying between New Orleans and Cuba, it is quite probable that the whole place may in a few years be materially changed, and by the clearing away of the old original "bush" the flora may undergo complete alteration.

It will be observed that the flora is completely Caribbean, and presents hardly any connexion with that of the mainland of Florida, with the exception of that small portion south of the "Everglades."

[* denotes that the plant is not originally indigenous.]

Papaveraceæ.

1. *Argemone Mexicana* (L.). Native, according to Chapman, and exceedingly abundant in most places.

Cruciferæ.

2. *Lepidium Virginicum* (L.). A more slender form than the ordinary plant. Common.

3. *Cakile æqualis* (L'Her.). Shifting sands of the south coast, Key West. Differs from *C. maritima* (L.) in the shape of the upper fruit-joint.

Capparidaceæ.

4. **Gynandropsis pentaphylla* (DC.). Occasionally in waste places.

5. *Capparis Jamaicensis* (Jay). Rare (*Mr. Féay*).

6. *Capparis Cynophallophora* (L.). Occasional (*Mr. Féay*).

Portulacaceæ.

7. *Portulaca oleracea* (L.). Everywhere, especially on paths and clearings in the bush. Flower yellow.

8. *Portulaca pilosa* (L.). Rare. Flowers purple.

Malvaceæ.

9. *Sida rhombifolia* (L.).
10. *Sida stipulata* (Cav.).
} Everywhere, in many forms.

[I did not observe *S. ciliaris* (Car.) and *S. Lindheimeri* (Gray), noted in Chapman's 'Flora,' p. 55, as occurring at Key West.]

11. *Abutilon crispum* (Gray). Not common.

12. *Hibiscus Floridanus* (Shuttl.).

13. **Hibiscus Rosa Sinensis* (L.). Escape from cultivation.

14. **Gossypium Barbadense* (L.). Large shrubs occur towards the Lighthouse, S.W. corner of Key West.

[*Ayenia pusilla* (W.), nat. ord. Byttneriaceæ, has occurred.]

Tiliaceæ.

15. *Corchorus siliquosus* (L.).

Olacaceæ.

16. *Ximenia Americana* (L.). (*Mr. W. T. Féay.*)

Aurantiaceæ.

17. **Citrus Aurantium* (L.). Very abundant in the S.W. quarter of the island, though, no doubt, originally imported.

Meliaceæ.

18. **Melia Azederach* (L.). The Pride of India is planted in nearly all villages and towns in Florida.

Oxalidaceæ.

19. *Oxalis stricta* (L.). Everywhere.

Zygophyllaceæ.

20. *Tribulus cistoïdes* (L.). Superficially resembling *Potentilla anserina* (L.).

[*Kallströmia maxima,* Torrey and Gray, was not observed, though it has occurred at Key West.]

21. *Guiacum sanctum* (L.). Southern portion of the island. Not common.

RUTACEÆ.

22. *Zanthoxylum Pterota* (H. B. & K.)—*Fagara lentiscifolia* (W.). One of the most abundant shrubs of the island; but impossible to preserve for herbarium purposes, as the joints of the petiole disintegrate immediately when dry.

SIMARUBACEÆ.

23. *Simaruba glauca* (DC.). One of the most conspicuous trees of the island, with large and handsome pinnate leaves and panicles of small green flowers.

BURSERACEÆ.

24. *Bursera gummifera* (Jay). A large tree (*Mr. Féay*).

25. *Amyris Floridana* (Nutt.).

ANACARDIACEÆ.

26. *Rhus Metopium* (L.). A good-sized tree; common.

VITACEÆ.

27. *Vitis (Cissus) acida* (L.). Not uncommon in the S.W. portion of the island. It disintegrates in drying, as *Z. Pterota.*

RHAMNACEÆ.

28. *Scutea ferrea* (Brong.). (*Mr. W. T. Féay.*) = *Condalia* (Cav.).

29. *Gouania Domingensis* (L.). Common.

CELASTRACEÆ.

30. *Schœfferia frutescens* (Jacq.). Abundant.

31. *Maytenus phyllanthoides* (Benth.). Leaves very fleshy. Common in salt-marshes.

LEGUMINOSÆ.

32. *Indigofera leptosepala* (Nutt.). (*Mr. W. Féay.*)

33. *Galactia spiciformis* (Torr. and Gray). Common.

34. *Piscidia erythrina* (L.). The Jamaica Dogwood. Not common.

35. *Cassia occidentalis* (L.). Waste places.

36. *Cassia biflora* (L.).

37. **Parkinsonia aculeata* (L.). Not mentioned in Chapman's 'Flora,' and possibly a recent introduction; but I obtained it from a remote quarter of the island where there was no cultivation. It is a remarkably elegant tree with spikes of orange-yellow flowers.

37 A. **Tamarindus indicus* (L.). Quite naturalized and common.

38. **Poinciana pulcherrima* (L.). Near the town of Key West. Now referred by most authors to the genus *Cæsalpinia.*

39. *Pithecolobium Unguis Cati* (Benth.). Very abundant by the sea in the south portion of the island.

40. *Pithecolobium Guadalupense* (Desv.). Not so frequent as the last, of which it is probably a variety.

41. **Acacia* (*Albizzia*) *Julibrizzin* (W.). Not unfrequent; naturalized.

42. **Acacia (Vachellia) Farnesiana* (W.). This is the most abundant shrub in the western portion of the island. Not being mentioned in Chapman's 'Flora' must surely be an error.

43. *Desmanthus diffusus* (Willd.). North shore of Key West. A prostrate form.

44. *Guillandina Bonducella* (L.). Very abundant on sandy ground near the South Fort. Not recorded in Chapman's 'Flora,' though apparently wild.

Myrtaceæ.

45. *Eugenia monticola* (DC.).
46. —— *buxifolia* (Willd.).
47. —— *procera* (Poir.).

} Abundant throughout the island.

48. *Calyptranthes Chytraculia* (Swartz). Not uncommon.

Rhizophoraceæ.

49. *Rhizophora Mangle* (L.). Mangrove. With *Avicennia oblongifolia* on the north shore of Key West.

Combretaceæ.

50. *Conocarpus erecta* (Jacq.). On sand by the south shore. Leaves remarkably white and silky.

51. *Terminalia Catappa* (L.). Very abundant.

52. *Laguncularia racemosa* (Gærtn.). North shore (*Mr. W. T. Féay*).

Cactaceæ.

53. *Cereus monoclonos* (DC.). Very conspicuous from its tall, column-like stems, 10 to 12 feet high. It is used, with *Agave Americana* and *Opuntia polyantha,* for hedges, and the three form an impenetrable barrier.

54. *Opuntia vulgaris* (L.), var. *polyantha*. Abundant everywhere.

PASSIFLORACEÆ.

55. *Passiflora angustifolia* (Sw.). Not common (*Mr. Féay*).

CUCURBITACEÆ.

56. *Sicyos angulatus* (L.).

CRASSULACEÆ.

57. **Bryophyllum calycinum* (L.). Very abundant. Not included in Chapman's 'Flora.'

SURIANACEÆ.

58. *Suriana maritima* (L.). Very abundant on the south shores of the island.

RUBIACEÆ.

59. *Spermacoce tenuior* (L.). Common in waste places.

60. *Ernodea littoralis* (S. W.). Not uncommon; flowers sweet-scented.

61. *Morinda Roioc* (L.).

62. *Chiococca racemosa* (Jacq.).

63. *Hamelia patens* (Jacq.). A very handsome shrub, with scarlet flowers. Western shores of the island.

64. *Randia aculeata* (L.). (*Mr. W. T. Féay.*)

65. *Exostemma Caribbæum* (R. & S.). (*Mr. W. T. Féay.*)

[*Erithalis fruticosa* (L.) has been found also at Key West.]

[*Guettarda elliptica* (S. W.). Key West; Rügel, in Herb. Shuttleworth, Mus. Brit.]

Compositæ.

66. *Cœlestina maritima* (Torr. & Gray). [*Ageratum*, L.] South shores of Key West. Abundant.

67. *Parthenium Hysterophorus* (L.). By roadsides, very abundant.

68. *Iva imbricata* (Walt.). Sandy shores.

69. *Ambrosia crithmifolia* (DC.). Very abundant along the southern shores.

70. *Borrichia arborescens* (DC.). Salt-marshes; abundant.

70 A. *Borrichia frutescens* (DC.). Not so common as the last.

[*Cosmos caudatus* (Kunth) occurs at Key West; but was not observed.]

71. *Bidens leucantha* (Willd.). A common tropical weed.

72. *Pluchea purpurascens* (DC.). Very common.

73. **Verbesina* (*Ximenesia*) *encelioides* (DC.). Abundant. Not included in Chapman's 'Flora.'

74. *Flaveria linearis* (Jay). Common.

75. *Sonchus oleraceus* (L.). Southern shores of Key West.

Sapotaceæ.

76. *Mimusops Sieberi* (A. DC.). =*M. dissecta* (R. Br.). Very abundant; a handsome tree. South shores of Key West.

Theophrastaceæ.

77. *Jacquinia armillaris* (Jacq.). (*Mr. W. T. Féay.*)

Myrsinaceæ.

78. *Ardisia Pickeringia* (Torr. and Gray). A fine shrub or small tree, flowering conspicuously in March.

Plumbaginaceæ.

79. *Plumbago scandens* (L.). Amongst *Opuntia,* in dry stony places; very abundant. Flowers white.

Bignoniaceæ.

80. *Tecoma stans* (Jussieu). Rare. Flowers very large, golden yellow.

Scrophulariaceæ.

81. *Herpestis peduncularis* (Benth.). Not uncommon. Flowers small, yellow. Turns quite black in drying, in common with most members of this family.

82. *Capraria biflora* (L.). Very common. Flowers varying from rose-pink to white.

Acanthaceæ.

83. *Dipteracanthus linearis* (Torr. and Gray).

84. *Dicliptera assurgens* (Juss.) = *D. sexangularis* (L.). S.W. of Key West, among cacti. Flowers scarlet.

Verbenaceæ.

85. *Priva echinata* (Juss.).

86. *Stachytarpheta Jamaicensis* (Vahl). Exceedingly abundant all round the coast. The flowers bright blue, in linear spikes.

87. *Lippia* (*Zapania*) *nodiflora* (Michx.). Very common.

88. *Lantana involucrata* (L.)., var. *Floridana*. The most frequent shrub on the island. Chapman ('Flora S. States,' p. 308) queries the colour of the corolla; it is white with a purplish tinge in the tube.

89. *Citharexylum villosum* (L.).

90. *Avicennia oblongifolia* (Nutt.). Common with the mangrove in swamps.

Labiatæ.

91. *Ocimum Campeachianum* (Mill.). Abundant; flowering in January and February.

92. *Salvia serotina* (L.). Abundant.

93. *Leonotis nepetæfolia* (R. Br.). Rare in the S.W. portion of the island.

Borraginaceæ.

94. *Cordia bullata* (L.). Rare. (*Mr. W. T. Féay.*)

95. **Cordia Sebestena* (L.). Probably not native. Near the town of Key West.

96. *Ehretia Buerreria* (L.). In the north part of the island; but not common.

97. *Ehretia tomentosa* (G. Don), var. *Havanensis* (W.). Very rare; one bush only observed in the central part of the island. Not in Chapman's 'Flora,' though Grisebach mentions its occurrence.

98. *Tournefortia Gnaphalodes* (R. Br.). By the seashore; very abundant.

99. *Tournefortia volubilis* (L.). Climbing up trees in the north portion of the island.

100. *Heliotropium Curassavicum* (L.). Salt-marshes. Common.

101. *Heliotropium myosotoïdes* (Chapman). Peculiar, I believe, to this one locality.

102. *Heliophytum parviflorum* (DC.). Abundant.

Convolvulaceæ.

103. *Pharbitis hispida* (Chois.). Very abundant as a climber in the north portion of the island. Flowers deep purple-blue, very showy.

104. *Ipomœa Pes-Capræ* (Sweet). Western and southern shores of the island. Very common. Found in all tropical countries on the sand of the seashore.

105. *Ipomœa sagittifolia* (B. R.). (*Mr. W. T. Féay.*) Rare.

106. *Ipomœa triloba* (L.). Not frequent.

107. *Ipomœa Bona nox* (L.). Abundant but local; flowering in the evening. One of the most beautiful climbing plants known, its pure white corolla being salver-shaped and perfectly flat, the tube being extremely long and pale greenish white. The flower fades at dawn.

108. *Jacquemontia violacea* (Chois.). Flowers small, sky-blue. Twining over shrubs, principally *Lantana*. Common in the south portion of Key West.

109. *Dichondra repens* (Forst.).

Solanaceæ.

110. *Solanum nigrum* (L.). A small-leaved form.

111. *Solanum verbascifolium* (L.). A tall shrubby plant with felted leaves and white flowers. Very common in the S.W. region.

112. *Solanum Bahamense* (L.).

113. *Solanum Blodgettii* (Chapman).

114. **Solanum Lycopersicum* (L.). Tomato. Waste places.

115. *Capsicum frutescens* (L.). Not uncommon.

116. *Physalis pubescens* (L.).

117. *Physalis angulata* (L.).

118. *Lycium Carolinianum* (Michaux). Abundant in the salt marshes.

119. **Cestrum fastigiatum* (Jacq.). Very common all round the S.W. portion of the island. Not included in Chapman's 'Flora.'

120. *Datula Tatula* (W.). A weed.

GENTIANACEÆ.

121. *Eustoma exaltatum* (Griseb.). By the battery, north shore. Three feet high; flowers very dark blue. A very beautiful plant.

APOCYNACEÆ.

122. *Echites umbellata* (Jacq.). Not uncommon.

123. *Echites Andrewsii* (Chapman). (*Mr. W. T. Féay.*) Rare.

124. *Vinca rosea* (L.). All round the island; in waste places, with *Ricinus, Ambrosia,* and *Argemone Mexicana.*

125. *Vallesia chioccoïdes* (Kunth) = *V. glabra* (Cav.). Very rare; only one shrub observed.

126. **Thevetia neriifolia* (Juss.) = *Cerbera Thevetia* (L.). Naturalized near the town. Exceedingly poisonous (*cf.* Kingsley's 'At Last' for a description).

Asclepiadeæ.

127. *Asclepias Curassavica* (L.). Waste places.

128. *Metastelma Schlectendalii* (Dec.). A twining creeper.

129. *Seutera maritima* (Reich.). By the salt-pans. Very abundant in one place only.

130. *Cyncotonum scoparium* (Meyer).

131. *Sarcostemma crassifolium* (Dur.). (*Mr. W. T. Féay.*)

Nyctagineæ.

132. **Mirabilis Jalapa* (L.). In one place towards the north shore; very likely an outcast from cultivation.

133. *Boerhaavia viscosa* (Lag.). Common.

134. *Pisonia aculeata* (L.). Exceedingly abundant, its greenish flowers proving very attractive to insect-life. The thorns on the branches and the recurved spines of the fruit are great impediments to comfort in the "bush."

Phytolaccaceæ.

135. *Rivina humilis* (L.). Common, and striking from its scarlet fruit.

136. **Phytolacca decandra* (L.). Occasionally on waste ground. Migrated from the Southern States of America.

Chenopodiaceæ.

137. **Chenopodium Anthelminticum* (L.). Naturalized from the States; one plant.

138. *Obione arenaria* (Moquin).

139. *Chenopodina maritima* (Moquin).

140. *Salicornia ambigua* (Michaux). By the salt-pans, with the preceding.

AMARANTACEÆ.

141. *Celosia paniculata* (L.). West shore. Common.

142. **Amarantus hybridus* (L.). Migrated from U. S. A.

143. **Amarantus albus* (L.). Migrated from U. S. A.

144. *Amarantus spinosus* (L.).

145. *Iresine vermicularis* (Moquin).

146. *Alternanthera Achyrantha* (R. Br.).

147. *Telanthera Floridana* (Chapman).

POLYGONACEÆ.

148. *Coccoloba uvifera* (Jay). South shores. Common. "The Sea Grape."

EUPHORBIACEÆ.

149. *Euphorbia cyathophora* (Jay), var. *graminifolia* (Michx.) = *E. heterophylla* (L.). Abundant in many places. Easily recognized by the uppermost leaves being deep scarlet at the base, and bearing, therefore, some slight resemblance to a small and narrow-leaved *Poinsettia*.

150. *Euphorbia glabella* (Swartz).

151. *Euphorbia hypericifolia* (L.). Sea-shore, in sand; common.

152. *Euphorbia maculata* (L.). Abundant in paths and everywhere.

153. *Euphorbia inæquilatera* (Sond.).

154. *Hippomane Mancinella* (L.). (*Mr. W. T. Féay.*)

155. *Acalypha corchorifolia* (Willd.).

156. *Croton balsamiferum* (Willd.). Very common in the south portion of the island only.

157. *Aphora Blodgettii* (Torrey). (*Mr. W. T. Féay.*)

158. **Ricinus communis* (L.). Everywhere.

Batidaceæ.

159. *Batis maritima* (L.). Salt-marshes.

Urticaceæ.

160. *Pilea hernarioïdes* (Lindley). Very abundant. A very small fragile plant.

Palmæ.

161. **Cocos nucifera* (L.). Naturalized in the northern portion of the island; common. The True Cocoanut Palm.

Potomaceæ.

162. *Halophila ovalis* (Hook.), or allied species. Quite fresh specimens floating in sea, after gale, 15 March, 1872.

Alismaceæ.

162 A. *Echinocarpus radiatus* (L.). Around a small pond in the interior of the island.

Orchidaceæ.

163. *Epidendrum venosum* (Lindley). North part of the island, on trees (*Mr. W. T. Féay*).

Amaryllidaceæ.

164. **Agave Americana* (L.). Forming impenetrable hedges in the north portion of the island.

Bromeliaceæ.

165. *Tillandsia bulbosa* (Hooker). (*Mr. W. T. Féay.*)

166. **Ananassa sativa* (Lindley). Occasionally escapes from cultivation.

Musaceæ.

167. **Musa sapientum* (L.). The Banana is extensively planted, and is sometimes found apparently naturalized.

168. **M. Paradisiaca* (L.). Ditto.

Cyperaceæ.

Of this order I found 5 species of *Cyperus*, including *C. confertus* (S.W.) and *C. fuligineus* (Chapm.), the others not yet determined, as they are in fragmentary condition, and also an *Eleocharis*, sp. incert.

Of Gramineæ about 10 to 12 species, including *Eragrostis ciliaris* (Link.); *Panicum* 2 species; the abundant *Dactyloctenium Ægyptiacum* (Willd.) and *Eleusine Indica* (Gærtn.) and *Cenchrus tribuloïdes* (L.). The curious creeping *Monanthochlöe littoralis* (Engelmann) also occurred on the southern sandy shores.

Only one Fern, that being *Aspidium patens* (Sw.); but Mr. Féay noted *Acrostichum aureum* (L.) and *Anemia adiantifolia* (L.) as well.

A *Chara* occurred plentifully in a pond towards the south portion of the island.

The whole flora of the island may therefore be summed up as not containing less than two hundred to two hundred and twenty species, of which perhaps one hundred and ninety may be considered genuine notices.

XIV. *On the Occurrence of Caffeine in the Leaves of Tea and Coffee grown at Kew Gardens.* By C. SCHORLEMMER, F.R.S.

Read April 3rd, 1883.

SOME time ago I wanted some information on the plants containing caffeine. This interesting compound is not only found in the seeds of *Coffea arabica,* but also in the fleshy part of the berry and in the leaves. The latter are used in Sumatra by the natives in the place of tea; and some years ago a patent was taken out for the introduction of "coffee-tea" into this country, but it has not been successful.

Caffeine also occurs in the leaves of different species or varieties of tea—in Paraguay tea, consisting of the leaves and small twigs of *Ilex paraguayensis,* and in Guarana or Brazilian chocolate, which is obtained from the roasted seeds of *Paullinia sorbilis.* It has further been found in cola-nuts, the seeds of *Sterculia acuminata,* which are extensively used as a condiment by the natives of western and central tropical Africa, and likewise by the negroes in the West Indies and Brazil, by whom the tree has been introduced into these countries. It is said that they promote digestion, improve the flavour of anything eaten after them, counteract the effects of alcohol, and even render half putrid water drinkable. Cola-nuts contain also theobromine, which is a very interesting fact, inasmuch as this base, which was first found in *Theobroma*

Cacao, which also belongs to the Sterculaceæ, can be easily transformed into caffeine.

In collecting this information I consulted, amongst other works, 'The Treasury of Botany,' edited by Lindley and Moore, where I found the statement that the custom of drinking coffee originated with the Abyssinians, who cultivated the plant from time immemorial. In Arabia it was not introduced until the early part of the fifteenth century; before this time the beverage made from the leaves of the kát (*Catha edulis*) was generally used, and is still in use, possessing properties resembling those of strong green tea, only more pleasing and agreeable. The leaves are also chewed, and are said to have the effect of producing great hilarity of spirits, and such an agreeable state of wakefulness that the Arabs, who chew them, are able to stand sentry all night long without feeling drowsy.

As the properties of kát resemble so much those of tea, it appeared to me highly probable that it contains caffeine. My friend Mr. H. Marshall Ward was kind enough to apply to Professor W. T. Thiselton Dyer, F.R.S., who supplied me in the beginning of December with fresh leaves of a plant growing in the temperate house at Kew Gardens. He also sent me a sample from the museum. He says, "The material in our museum is not very satisfactory; but I enclose a portion of an authentic sample. The leaves are different in form from those of the living plant; but I have ascertained that in this respect they are variable."

I first examined the fresh leaves; not a trace of caffeine could be found, while its presence can be easily shown in three or four tea-leaves. The only crystalline compound which I could extract from *Catha* is a kind of sugar, apparently mannite. The quantity was, however, too small for identifying it.

No caffeine could be detected in the authentic sample, but it contained common salt, showing that it must have been in contact with sea-water.

As I fully expected to find caffeine, and was disappointed, it occurred to me that possibly the leaves of tea and coffee grown at Kew might also not contain their characteristic principle.

Professor Dyer kindly supplied me with the fresh leaves of several varieties and species in the middle of February.

I found caffeine in green tea (*Thea viridis*) and in Assam tea (*T. assamica*).

The leaves of *Coffea arabica*, which is now fruiting, contain it also, but much less in proportion than tea.

In *Coffea laurina* I found a trace, but none at all in the large, old leaves of Liberian coffee.

Caffeine and theobromine occur only in the vegetable kingdom. They are nearly related to uric acid, guanine, and xanthin, which are products of the material exchanges of the animal organism. The two former can be converted into xanthin, and this, as Professor E. Fischer has shown, into theobromine or methylxanthin, and into caffeine or dimethylxanthin.

As caffeine is used in medicine, it appears very probable that at no distant time it will be manufactured from Peruvian guano, which is the best source for guanine.

XV. *On the Change produced in the Motion of an Oscillating Rod by a Heavy Ring surrounding it and attached to it by Elastic Cords.* By JAMES BOTTOMLEY, B.A., D.Sc., F.C.S.

Read October 2nd, 1883.

AT a meeting of the Mathematical Section on January 16th Dr. Joule brought under the notice of the members a method which he had devised for damping the small oscillations of a telescope, or any other heavy mass which it is desirable to keep as steady as possible. The arrangement consisted of a heavy ring surrounding the vibrating mass and attached to it by extensible strings.

Possibly a telescope may not vibrate as a single mass; for the parts of which it is composed may not be so rigidly connected as to prevent small relative motions. For simplicity I have considered a case where the period and amplitude of the oscillation may be assigned. Suppose we have a heavy rod supported from a fixed beam by two elastic cords of given length, and attached to the rod at points equidistant from the middle. Suppose an impulse to be given to the rod in the plane of the cords; then it will execute harmonic motions. Now suppose a single damper placed at the middle of the rod, how will this affect its motion?

Let m denote the mass of the rod, m_1 mass of the ring, λ modulus of elasticity of cords connecting the rod with the ring, λ' modulus of elasticity of cords supporting the rod, D diameter of ring, d diameter of rod.

The ring exhibited by Dr. Joule had three cords; to simplify the solution I suppose that there are only two, both in the plane of motion, also I suppose that the mass of these cords is so small that the tension due to their own weight may be neglected, and that the cords are uniformly stretched.

To determine the position of rest let T denote the tension in each of the cords supporting the rod, t_1 the tension in the upper cord of the damper, t_2 the tension in the lower cord; then

$$mg - 2T - t_1 + t_2 = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$m_1 g + t_1 - t_2 \qquad = 0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let L be the unstretched length of cords supporting the rod, L_1 the stretched length, l the unstretched length of the cords attached to the ring, l_1 the stretched length of the upper cord, l_2 the stretched length of the lower cord; then

$$T = \frac{\lambda'}{L}(L_1 - L), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$t_1 = \frac{\lambda}{l}(l_1 - l), \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$t_2 = \frac{\lambda}{l}(l_2 - l). \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Also we have the relation

$$l_1 + l_2 = D - d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Let L_2 be the distance of the centre of the ring from the fixed beam; then we have

$$l_1 = L_1 - L_2 + \frac{D}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$l_2 = L_2 - L_1 + \frac{D}{2} - d. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From the foregoing equations we obtain

$$L_1 = L + g(m+m_1)\frac{L}{2\lambda'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$L_2 = g(m+m_1)\frac{L}{2\lambda'} + L + \frac{d}{2} + m_1 g\frac{l}{2\lambda}. \quad . \quad . \quad (10)$$

If L_3 be the distance of the axis of the rod from the fixed beam, we have

$$L_3 = L_1 + \frac{d}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If we add (4) and (5), and substitute from (6), we get

$$t_1 + t_2 = \frac{\lambda}{l}(D - d - 2l). \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Thus the sum of the tensions is constant, and is independent of the weight of the ring.

If the following condition holds, there will be no tension in the upper cord,

$$l = \lambda \,.\, \frac{D-d}{m_1 g + 2\lambda}.$$

Now suppose the rod disturbed from its position of rest. Let X denote the distance of the axis of the rod from the fixed beam, and X_1 the distance of the centre of the ring. The tension in each cord sustaining the rod will be

$$T = \frac{\lambda'}{L}\left(X - \frac{d}{2} - L\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

also

$$t_1 = \frac{\lambda}{l}\left(X - X_1 + \frac{D-d}{2} - l\right), \quad . \quad . \quad . \quad (14)$$

$$t_2 = \frac{\lambda}{l}\left(X_1 - X + \frac{D-d}{2}\right) - l. \quad . \quad . \quad . \quad (15)$$

The equations of motion are, for the rod,

$$m\frac{d^2X}{dt^2}=mg-2T+t_2-t_1,$$

and for the ring

$$m_1\frac{d^2X_1}{dt^2}=m_1g-t_2+t_1.$$

Substituting from (13, 14, 15), we have

$$m\frac{d^2X}{dt^2}=mg-2\frac{\lambda'}{L}\left(X-\frac{d}{2}-L\right)+2\frac{\lambda}{l}(X_1-X), \quad . \quad (16)$$

and for the ring

$$m_1\frac{d^2X_1}{dt^2}=m_1g-2\frac{\lambda}{l}(X_1-X). \quad . \quad . \quad (17)$$

Let x denote the displacement of the axis of the rod from its position of rest, and x_1 the displacement of the centre of the ring from its position of rest; then

$$X=x+L_1+\frac{d}{2},$$

$$X_1=x_1+L_2.$$

Substituting in equations (16) and (17), and cancelling those terms which are in equilibrium by the statical equations, we obtain

$$m\frac{d^2x}{dt^2}=-2\frac{\lambda'}{L}x-2\frac{\lambda}{l}x+2\frac{\lambda}{l}x_1, \quad . \quad . \quad . \quad (18)$$

$$m_1\frac{d^2x_1}{dt^2}=\quad 2\frac{\lambda}{l}x-2\frac{\lambda}{l}x_1; \quad . \quad . \quad . \quad . \quad . \quad (19)$$

by addition we have

$$m\frac{d^2x}{dt^2}+m_1\frac{d^2x_1}{dt^2}=-2\frac{\lambda'}{L}x. \quad . \quad . \quad . \quad (20)$$

Differentiating (18) twice,

$$m\frac{d^4x}{dt^4} = -2\frac{d^2x}{dt^2}\left(\frac{\lambda'}{L}+\frac{\lambda}{l}\right)+2\frac{\lambda}{l}\frac{d^2x_1}{dt^2}; \quad . \quad . \quad . \quad (21)$$

substituting from (20), we have

$$\frac{d^4x}{dt^4}+A\frac{d^2x}{dt^2}+Bx=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where

$$A=\frac{2}{m}\left(\frac{\lambda'}{L}+\frac{\lambda}{l}+\frac{\lambda m}{lm_1}\right),$$

and

$$B=4\frac{\lambda'}{L}\frac{\lambda}{lmm_1}.$$

To solve the above equation, assume

$$x=Ce^{\mu t}.$$

Differentiating this twice and four times, substituting in (22), and dividing by a common factor, we have

$$\mu^4+A\mu^2+B=0.$$

Hence

$$\mu^2=-\frac{A}{2}\pm\sqrt{\frac{A^2}{4}-B}.$$

The quantity under the radical sign will be positive, zero, or negative according as

$$\left(\frac{\lambda'}{L}+\frac{\lambda}{l}+\frac{\lambda m}{lm_1}\right)^2 > = < 4\frac{\lambda'}{L}\frac{\lambda m}{lm_1},$$

according as

$$\left(\frac{\lambda'}{L}+\frac{\lambda}{l}-\frac{\lambda}{l}\frac{m}{m_1}\right)^2+4\frac{m}{m_1}\frac{\lambda^2}{l^2} > = < 0.$$

The expression on the left is essentially positive; therefore the quantity under the radical sign is positive, and

$$\sqrt{1-\frac{B}{\frac{A^2}{4}}}$$

will be a fractional quantity. Let it be denoted by e, so that

$$\mu^2 = -\frac{A}{2}(1 \mp e).$$

Hence, the four roots of the biquadrate equation are

$$\sqrt{-1}\sqrt{\frac{A}{2}(1-e)}, \qquad \sqrt{-1}\sqrt{\frac{A}{2}(1+e)},$$

$$-\sqrt{-1}\sqrt{\frac{A}{2}(1-e)}, \quad -\sqrt{-1}\sqrt{\frac{A}{2}(1+e)},$$

and are all imaginary. Writing p for $\sqrt{\frac{A}{2}(1-e)}$, and q for $\sqrt{\frac{A}{2}(1+e)}$, we obtain as the general solution of equation (22),

$$x = c_1\epsilon^{pt\sqrt{-1}} + c_2\epsilon^{-pt\sqrt{-1}} + c_3\epsilon^{qt\sqrt{-1}} + c_4\epsilon^{-qt\sqrt{-1}},$$

or if we substitute angular functions for the impossible exponentials,

$$x = c_1(\cos pt + \sqrt{-1}\sin pt) + c_2(\cos pt - \sqrt{-1}\sin pt)$$
$$+ c_3(\cos qt + \sqrt{-1}\sin qt) + c_4(\cos qt - \sqrt{-1}\sin qt).$$

This may be more briefly written

$$x = P\cos pt + Q\cos qt + R\sin pt + S\sin qt.$$

P, Q, R, S may be determined by the initial conditions of the motion. When $t=0$, $x=0$. Hence we have

$$0 = P + Q. \quad \ldots \ldots \quad (23)$$

Differentiate, put $t=0$, and V for the initial velocity,

$$V = Rp + Sq. \quad \ldots \ldots \quad (24)$$

Differentiating twice, and substituting initial values, we have

$$0 = -Pp^2 - Qq^2; \quad \ldots \ldots \quad (25)$$

whence, by (23),

$$0=P(q^2-p^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

Since q^2-p^2 cannot vanish, the only solution of this equation is $P=0$. Therefore also

$$Q=0,$$

and the equation becomes

$$x=R\sin pt+S\sin qt.$$

To determine the two constants R and S we have only one equation (24). To obtain a second equation we must know the initial velocity of the ring. This is quite arbitrary, for evidently, when the rod is set in motion, we may simultaneously impress any velocity we please upon the ring. Suppose that the rod is set in motion by a blow and that the ring is initially at rest. Differentiating the last equation three times we get

$$\frac{d^3x}{dt^3}=-Rp^3\cos pt-Sq^3\cos qt.$$

Differentiate (18), substitute for x and $\frac{d^3x}{dt^3}$ from the two last equations, make $\frac{dx_1}{dt}=0$, and put initial values for the other quantities; then we get this equation,

$$m(Rp^3+Sq^3)=2V\left(\frac{\lambda'}{L}+\frac{\lambda}{l}\right).$$

From this equation, combined with (24), we derive the following values:—

$$R=\frac{V}{mp(q^2-p^2)}\left\{mq^2-2\left(\frac{\lambda'}{L}+\frac{\lambda}{l}\right)\right\},$$

$$S=-\frac{V}{mq(q^2-p^2)}\left\{mp^2-2\left(\frac{\lambda'}{L}+\frac{\lambda}{l}\right)\right\}.$$

Hence the equation of motion of the rod takes the form

$$x = \frac{V}{m(q^2 - p^2)} \cdot \left\{ \left(mq^2 - 2\left(\frac{\lambda'}{L} + \frac{\lambda}{l}\right)\right) \frac{\sin pt}{p} \right.$$
$$\left. - \left(mp^2 - 2\left(\frac{\lambda'}{L} + \frac{\lambda}{l}\right)\right) \frac{\sin qt}{q} \right\}.$$

Since q is greater than p, and $mq^2 - 2\left(\frac{\lambda'}{L} + \frac{\lambda}{l}\right)$ is positive, and $mp^2 - 2\left(\frac{\lambda'}{L} + \frac{\lambda}{l}\right)$ is negative, it follows that both R and S are positive quantities. The motion of the rod is therefore made up of two harmonic motions of unequal periods and unequal amplitudes.

To determine the motion of the ring differentiate the last equation twice, and substitute in (18); we then obtain

$$x_1 = \frac{Vl}{2\lambda m(q^2 - p^2)} \left(mq^2 - 2\left(\frac{\lambda'}{L} + \frac{\lambda}{l}\right)\right) \left(2\left(\frac{\lambda'}{L} + \frac{\lambda}{l}\right) - mp^2\right)$$
$$\left(\frac{\sin pt}{p} - \frac{\sin qt}{q}\right).$$

By some substitutions this equation may be represented in the following simpler form :—

$$x_1 = \frac{2Vl}{m_1 l(q^2 - p^2)} \left(\frac{\sin pt}{p} - \frac{\sin qt}{q}\right).$$

If we differentiate this equation we find that the ring will be at rest whenever the following condition is fulfilled :—

$$\cos qt - \cos pt = 0.$$

This condition is satisfied when

$$t = \frac{2n\pi}{q + p},$$

and also when

$$t=\frac{2n\pi}{q-p},$$

n being any positive integer including o.

If among the physical constants we have the following relation

$$\frac{\lambda'}{L}+\frac{\lambda}{l}-\frac{\lambda m}{l m_1}=0,$$

the equation of motion of the rod takes the form

$$x=\frac{V}{2}\left(\frac{\sin pt}{p}+\frac{\sin qt}{q}\right),$$

and the equation of motion of the ring takes the form

$$x_1=\frac{V}{2\sqrt{mm_1}}\left(\frac{\sin pt}{p}-\frac{\sin qt}{q}\right).$$

The equation of motion of the rod before attaching the ring would be

$$x=\frac{V}{\sqrt{\frac{2\lambda'}{Lm}}}\cdot\sin\sqrt{\frac{2\lambda'}{Lm}}\cdot t,$$

and if the ring were rigidly attached to the rod, the equation of motion would be

$$x=\frac{mV}{\sqrt{m+m_1}}\cdot\sqrt{\frac{L}{2\lambda'}}\cdot\sin\sqrt{\frac{2\lambda'}{L(m+m_1)}}\cdot t,$$

the impulsive force setting the system in motion being supposed to be the same in each case.

Instead of a single ring situated at the middle of the rod, we might have a number of rings symmetrically disposed about the middle. Let x_1, x_2, &c. be the distances of the rings from their position of rest, and

suppose all the rings to have the same mass; then the equations of motion are

$$m\frac{d^2x}{dt^2}=-2\frac{\lambda'}{L}x-2\frac{\lambda}{l}(x-x_1)-2\frac{\lambda}{l}(x-x_2)-\&c., \quad (27)$$

$$m_1\frac{d^2x_1}{dt^2}=\quad 2\frac{\lambda}{l}(x-x_1),$$

$$m_1\frac{d^2x_2}{dt^2}=\quad 2\frac{\lambda}{l}(x-x_2),$$

$$m_1\frac{d^2x_n}{dt^2}=\quad 2\frac{\lambda}{l}(x-x_n).$$

By addition we obtain

$$m\frac{d^2x}{dt^2}+m_1\left(\frac{d^2x_1}{dt^2}+\frac{d^2x_2}{dt^2}\ \ldots\ldots\ \frac{d^2x_n}{dt^2}\right)=-2\frac{\lambda'}{L}x. \quad . \quad (28)$$

Differentiating (33) twice, we obtain

$$m\frac{d^4x}{dt^4}=-2\frac{d^2x}{dt^2}\left(\frac{\lambda'}{L}+\frac{n\lambda}{l}\right)+2\frac{\lambda}{l}\left(\frac{d^2x_1}{dt^2}+\frac{d^2x_2}{dt^2}\ \ldots\ldots\ \frac{d^2x_n}{dt^2}\right);$$

substituting from (34), we obtain an equation of the form

$$\frac{d^4x}{dt^4}+A\frac{d^2x}{dt^2}+Bx, \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where B is written for

$$4\frac{\lambda'}{L}\frac{\lambda}{lmm_1}$$

and A for

$$\frac{2}{m}\left(\frac{\lambda'}{L}+\frac{n\lambda}{l}+\frac{\lambda m}{lm_1}\right).$$

If we proceed to solve this equation in the same manner as equation (22), we obtain as the auxiliary equation,

$$\mu^4+A\mu^2+B.$$

In a similar manner it may be shown that the roots of this equation are all imaginary, for

$$\left(\frac{\lambda'}{L}+\frac{\lambda}{l}n-\frac{\lambda m}{lm_1}\right)^2+4n\frac{\lambda^2 m}{l^2 m_1}$$

is essentially positive. Hence the solution of (29) will take the form

$$x = P\cos pt + Q\cos qt + R\sin pt + S\sin qt,$$

where

$$p=\sqrt{\frac{A}{2}-\sqrt{\frac{A^2}{4}-B}},$$

and

$$q=\sqrt{\frac{A}{2}+\sqrt{\frac{A^2}{4}-B}}.$$

If we suppose the rod to start from a position of rest with velocity V, P and Q will be each 0, and the equation will take the form

$$x = R\sin pt + S\sin qt,$$

with the condition

$$V = Rp + Sq.$$

The other equation necessary to determine R and S is quite arbitrary, and depends upon what hypothesis we make regarding the initial velocities of the rings. The subsidence of motion due to internal friction in elastic solids has not been taken into account.

XVI. *On the Leaves of* Catha edulis.

By C. Schorlemmer, F.R.S.

Read October 16th, 1883.

In a paper which I read before the Society a few months ago, I stated that the custom of drinking coffee was introduced into Arabia only in the beginning of the fifteenth century. Before this time the beverage made of leaves of kát (*Catha edulis*) was used, and is still in use, possessing properties resembling those of strong green tea, only more pleasing and agreeable. From this it appeared to me highly probable that kát contained caffeine, which occurs in tea, coffee, and some other plants, all of which are used as stimulants.

Professor T. Thiselton Dyer, F.R.S., kindly supplied me with fresh leaves of *Catha*, grown in Kew Gardens. Not a trace of caffeine was found, while, to show its presence in tea, a very few leaves are sufficient.

From the facts it appeared probable that the tea and coffee grown at Kew would not contain it; that, however, was found not to be the case. Its presence could be easily detected in the leaves of *Thea viridis* and *Coffea arabica*.

Through the kindness of Professor Dyer I have since obtained two genuine samples of kát, coming directly from Aden, one being labelled "Kát Sabäri," so called as coming from Sabar, a mountain-range in Yemen, and the other "Kát Mactari Asháb," as it has long (asháb) leaves. I have examined both carefully without finding a trace of caffeine or an alkaloid related to it. It requires, therefore, further researches in order to discover the active principle of *Catha*.

XVII. *On Singular Solutions of Differential Equations.*

By Robert Rawson, F.R.A.S., Hon. Memb. Manchester Literary and Philosophical Society, Assoc. I. N. A., Memb. of the London Math. Society.

Read November 14, 1882.

1. Singular solutions of differential equations have received great attention by most mathematicians since Dr. Brook Taylor announced the first example of them in 1715.

This is not to be wondered at, as it is important to know that differential equations, even of the first order and of the nth degree, may have other solutions than those commonly called their complete primitives.

In the application of the higher mathematics to physical problems of practical importance there should arise a relation between two variables x and y, with an arbitrary constant expressed by the differential equation

$$\left(\frac{dy}{dx}\right)^2+2\left(\sqrt{x+y}+\sqrt{x-y}\right)\frac{dy}{dx}+4\sqrt{x^2-y^2}$$
$$+2\left(\sqrt{x-y}-\sqrt{x+y}\right)-1=0, \quad . \quad . \quad (a)$$

whose complete primitive is

$$c^2+\left(\sqrt{x+y}+\sqrt{x-y}\right)c+\sqrt{x^2-y^2}$$
$$+x\left(\sqrt{x-y}-\sqrt{x+y}\right)-x^2=0. \quad . \quad . \quad (b)$$

It must be important to know that (a) has two solutions, viz. $x+y=0$, and $x-y=0$, besides the complete primitive (b), neither of which is a particular case of (b).

In the determination of singular solutions it has been

the custom hitherto, following the example of Lagrange, to put in the place of the arbitrary constant a certain function of x, y, such as to give the same differential equation as that which is derived from the primitive before the change of the arbitrary constant is made.

In what follows a different procedure has been adopted, viz. to impress upon the complete primitive such a form as to produce two or more solutions of the derived differential equation, neither of which shall be a particular case of the complete primitive in the ordinary sense.

By the former method the *envelope species* only of singular solutions is obtainable, whereas by the latter the *non-envelope species* as well as the *envelope species* are readily developed.

At the Ordinary Meeting of this Society, March 21st, 1882, Sir James Cockle, F.R.S. &c., read an interesting paper "On Envelopes and Singular Solutions," in which he refers to two theorems which I communicated to him in 1867.

The theorems are correctly stated by Sir James Cockle, and are as follows:—

A. The condition of equal roots with respect to c, the arbitrary constant, in the complete primitive $\phi(x, y, c)=0$ is a singular solution, if not contained in the primitive by giving a constant value to (c).

B. The condition of equal roots with respect to $p=\dfrac{dy}{dx}$ in the differential equation $\psi(x, y, p)=0$ is a singular solution if it satisfies the differential equation.

These theorems, which facilitate the finding of singular solutions both from the complete primitive and its derived

differential equation, are unnoticed by Boole, Hymer, De Morgan, Abbé Moigno, and others.

It seems, therefore, only fair to conclude that the above distinguished writers on the solution of differential equations failed to see or perceive the theorems (A) and (B) in the incidental remarks made by Lagrange, and referred to by Sir James Cockle in paragraphs 9, 10, 11.

Had Lagrange himself perceived the force and generality of his incidental remark, no doubt he would have deduced the theorems (A) and (B), and would have applied them to the solutions of the numerous examples which are given in the 'Leçons sur le Calcul des Fonctions,' as being more easy of application than are $\frac{dy}{dc}$, $\frac{dx}{dc}$, $\frac{dp}{dy}$, $\frac{d}{dx}\left(\frac{1}{p}\right)$.

The two theorems (A) and (B) were communicated to my mathematical friends Sir James Cockle, Rev. Robert Harley, Mr. Woolhouse, Mr. Purkiss, who was senior wrangler in 1864, Dr. Hirst, and others many years ago. It is now upwards of twenty years since these theorems were explained and taught to my pupils in Her Majesty's Dockyard Schools, Portsmouth.

It seems to be necessary to enter somewhat into these particulars, as Professor Cayley, of Cambridge, has communicated two very comprehensive papers "On the Theory of the Singular Solutions of Differential Equations of the First Order" to the 'Messenger of Mathematics,' vol. ii. 1872, p. 6, and vol. vi. 1876, p. 23. In these papers Professor Cayley has remarked "that the discriminant of the differential equation in regard to (p), and that of the integral equation in regard to (c), are each equal (y^2-1), and we have a true singular solution (y^2-1)," Ex. 1, p. 24.

The same process is repeated in the examples 2, 3, 4, and it is scarcely necessary to remind mathematicians of the close proximity of this process to the two theorems (A) and (B).

Again, in the 'Messenger of Mathematics' for May 1882, J. W. L. Glaisher, M.A., F.R.S., has an interesting paper, including pages 1 to 13, entitled "Examples illustrative of Professor Cayley's Theory of Singular Solution." The theorems (A) and (B), or the condition of equal roots, are applied by Mr. Glaisher to the solution of twenty-one examples, and he states that "in my lectures on differential equations I have for some years illustrated Cayley's theory of singular solutions, of which the preceding paragraphs contain a short résumé, by several simple examples."

As, however, neither Professor Cayley nor Mr. Glaisher has given a proof of the two theorems in question, I will endeavour to supply a demonstration of them, and to show the range of their application to determine the singular solution both from the complete primitive and its derived differential equation.

2. It is well known that the most general differential equation of the first order and of the nth degree is

$$\left(\frac{dy}{dx}+r_1\right)\left(\frac{dy}{dx}+r_2\right)\ldots\left(\frac{dy}{dx}+r_n\right)=0, \quad . \quad . \quad (1)$$

where the roots with respect to $\left(\frac{dy}{dx}\right)$ are $r_1, r_2 \ldots r_n$ functions of x, y.

The complete primitive of (1), therefore, is an equation of the form

$$(c_1+R_1)(c_2+R_2)\ldots(c_n+R_n)=0, \quad . \quad . \quad (2)$$

where R_1, R_2 . . . R_n are functions of x, y, which must satisfy the following partial differential equations, viz.

$$\left.\begin{array}{l} \dfrac{dR_1}{dx} = r_1 \dfrac{dR_1}{dy} \\ \dfrac{dR_2}{dx} = r_2 \dfrac{dR_2}{dy} \\ \vdots \qquad\quad \vdots \\ \dfrac{dR_n}{dx} = r_n \dfrac{dR_n}{dy}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

And c_1, c_2 . . . . c_n are arbitrary independent constants, which, however, may be made to satisfy $(n-1)$-fold relations.

3. An interesting case of the above is where each of the values of c_1, c_2 . . . . c_n is represented by the arbitrary constant (c); then (2) may be written

$$(c+R_1)(c+R_2) \;.\;.\;.\;.\; (c+R_n)=0, \quad . \quad . \quad . \quad (4)$$

where R_1, R_2 . . . . R_n are the roots of the primitive (4) with respect to the arbitrary constant (c).

Equation (1) in its present form is not generally an exact differential equation of the primitive (4); the multiplier, however, which will make it exact is readily obtained by differentiating (4) with respect to x, and will be as follows :—

$$\frac{1}{\begin{array}{c}(R_1-R_2)^2 \ldots (R_1-R_n)^2 \times (R_2-R_3)^2 \ldots (R_2-R_n)^2 \\ \times (R_3-R_4)^2 \ldots (R_3-R_n)^2 \ldots \times (R_4-R_5)^2 \ldots (R_4-R_n)^2\end{array}} . \quad (5)$$

Now it is proved by writers on the subject of singular solutions of differential equations that a singular solution

makes the multiplier, or the integrating factor, infinite; hence from (5) the singular solutions of (1) are

$$(R_1-R_2)^2\ldots(R_1-R_n)^2\times(R_2-R_3)^2\ldots(R_2-R_n)^2$$
$$\times(R_3-R_4)^2\ldots(R_3-R_n)^2\ \&c.=0. \quad \ldots \quad (6)$$

Of course each of the factors must be equated to zero, therefore the theorem (A) is obvious.

From (3), if $R_1=R_2$ then will $r_1=r_2$ &c., therefore the truth of theorem B is manifest.

4. In consequence of the importance of the principle of the condition of equal roots in the determination of singular solutions of differential equations, another proof of it may not be deemed unnecessary.

In the primitive (4) the functions $R_1, R_2 \ldots R_n$ can be made to assume various forms; put, therefore,

$$R_1=v_1+\sqrt{w_1},\quad R_2=v_1-\sqrt{w_1}, \quad \ldots \quad (7)$$

$$R_3=v_2+\sqrt{w_2},\quad R_4=v_2-\sqrt{w_2}, \quad \ldots \quad (8)$$

$$R_5=v_3+\sqrt{w_3},\quad R_6=v_3-\sqrt{w_3}, \quad \ldots \quad (9)$$

$$\vdots \qquad\qquad \vdots$$

$$\&c., \qquad\qquad \&c.,$$

where v_1, w_1; v_2, w_2; v_3, w_3, &c. are functions of x, y.

The primitive (4) becomes by substitution

$$\{(c+v_1)^2-w_1\}\{(c+v_2)^2-w_2\}\{(c+v_3)^2-w_3\}\ldots\&c.=0, \quad . \quad (10)$$

where (c) in each quadratic factor has a two-fold value.

Differentiate (10) with respect to x, eliminate (c), and reject the common factors; then

$$\left\{\frac{dy}{dx}+\frac{\pm 2\sqrt{w_1}\dfrac{dv_1}{dx}-\dfrac{dw_1}{dx}}{\pm 2\sqrt{w_1}\dfrac{dv_1}{dy}-\dfrac{dw_1}{dy}}\right\}\left\{\frac{dy}{dx}+\frac{\pm 2\sqrt{w_2}\dfrac{dv_2}{dx}-\dfrac{dw_2}{dx}}{\pm 2\sqrt{w^2}\dfrac{dv_2}{dy}-\dfrac{dw_2}{dy}}\right\}$$
$$\ldots\ \&c.=0, \quad \ldots \quad (11)$$

which is a differential equation having for its complete primitive equation (10).

Equation (11) is not, however, an exact differential, as the multipliers which would make it so are

$$\frac{1}{\sqrt{w_1}}, \quad \frac{1}{\sqrt{w_2}} \ldots \&c.$$

The first, second, &c. factors of (11) are evidently satisfied by the relations

$$\left.\begin{array}{l} w_1 = 0, \\ w_2 = 0, \\ \&c. \quad \&c., \end{array}\right\} \quad \ldots\ldots \quad (12)$$

which are, therefore, singular solutions of (11) if they are not contained in the primitive (10) by giving a constant value to (c).

By this assumption of the values of R_1, R_2 ... &c., it is seen at once that the differences R_1-R_2, R_3-R_4 ... &c., are made to vanish by the conditional equations (12); hence the truth of theorem (A).

Since the equations (12) make the roots, with respect to (p) in the quartic factors which compose (11) equal, therefore the theorem (B) is also proved.

The singular solutions here considered are of the envelope species, as they may be obtained by eliminating (c) between (10) the complete primitive and $\frac{dy}{dc}=0$. It will be seen that the integrating factors $\frac{1}{\sqrt{w_1}}$, $\frac{1}{\sqrt{w_2}}$, &c. are made infinite by singular solutions, which is a well-known property.

5. In this paragraph there will be proved some of the properties of singular solutions by means of the condition of equal roots.

Differentiate (10) with respect to (c), reject the common factors; then

$$\left\{\frac{dy}{dc}+\frac{\pm 2\sqrt{w_1}}{\pm 2\sqrt{w_1}\frac{dv_1}{dy}-\frac{dw_1}{dy}}\right\}\left\{\frac{dy}{dc}+\frac{\pm 2\sqrt{w_2}}{\pm 2\sqrt{w_2}\frac{dv_2}{dy}-\frac{dw_2}{dy}}\right\}$$

$$\ldots\ \&c.=0. \quad . \quad . \quad . \quad (13)$$

Now the condition of equal roots gives $w_1=0$, $w_2=0$, &c., therefore $\frac{dy}{dc}$ in (13) is evidently zero. It may be shown in a similar manner that $\frac{dx}{dc}$ is zero also.

In the case of the envelope species of singular solutions $\frac{dy}{dc}=0$ and $\frac{dx}{dc}=0$ are equivalent. Boole states that such, "although not necessarily equivalent, do not lead to conflicting results" (Boole's Diff. Eqs. p. 146, 2nd ed.).

For a particular hypothesis with respect to the form of the complete primitive, viz.

$$\int Q(c-X)^m dc - y = 0, \quad . \quad . \quad . \quad . \quad (14)$$

where Q is a function of x, c, which neither vanishes nor becomes infinite when $c=X$, Boole has proved that for a singular solution $\left(\frac{dp}{dy}\right)=$infinity. Boole also states "that inquiries which are scarcely of a sufficiently elementary character to find a place in this work indicate (with very high probability) that this character is universal and independent of any particular hypothesis, and that it constitutes a criterion for distinguishing solutions of the envelope species from others" (Boole's Diff. Eqs. p. 163, 2nd ed.).

The proposition, viz. $\frac{dp}{dy}=$infinity, which seems to have originated with Laplace, and was subsequently investigated by Lagrange, Cauchy, Poisson (see Boole's Diff. Eqs. pp. 172, 174, 2nd ed.), may be proved generally as follows :—

Differentiate each factor in equation (11), and reject the common factors; there results for the first factor by dropping the affixes,

$$\frac{dp}{pdy}=\frac{\pm\left(\frac{dw}{dy}\frac{dv}{dx}\right)\frac{1}{\sqrt{w}}\pm 2\sqrt{w}\frac{d^2v}{dxdy}-\frac{d^2w}{dxdy}}{\pm 2\sqrt{w}\frac{dv}{dx}-\frac{dw}{dy}}$$

$$-\frac{\pm\left(\frac{dw}{dy}\frac{dv}{dy}\right)\frac{1}{\sqrt{w}}\pm 2\sqrt{w}\frac{d^2v}{dy^2}-\frac{d^2w}{dy^2}}{\pm 2\sqrt{w}\frac{dv}{dy}-\frac{dw}{dy}}. \quad . \quad (15)$$

For a singular solution $w=0$, then (15) becomes

$$\frac{dp}{dy}=\text{infinity}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If

$$\pm\left(\frac{dv}{dx}\frac{dw}{dy}-\frac{dv}{dy}\frac{dw}{dx}\right)\frac{1}{\frac{dw}{dy}}$$

is a finite quantity, and v a function of x, y. When, however, the above quantity is zero, then $\frac{dp}{dy}=\frac{0}{0}$, an indeterminate quantity, and by Boole's Diff. Eqs. p. 24, 2nd ed., v will be a function of w, and thereby it is probable that the solution will be a particular integral.

6. As the differential equation of the first order and of the nth degree is composed entirely of the simple factors of a differential equation of the first order and degree, it will be necessary to consider the latter form only.

Let

$$\frac{dy}{dx}+r=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

be a differential equation where r is a function of x, y.

Put

$$c+\mathrm{R}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

for the complete primitive of (17), where (c) is an arbitrary constant and (R) a function of x, y, which, however, must satisfy the partial differential equation

$$\frac{d\mathrm{R}}{dx}=r\frac{d\mathrm{R}}{dy}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

7. The function (R) may be taken so as to satisfy the equation

$$\mathrm{R}=v+zf(w), \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where v, z, w are functions of x, y.

Substitute this value in (17) and (18); they become respectively

$$\frac{dy}{dx}+\frac{\left\{\frac{dv}{dx}+\frac{dz}{dx}f(w)\right\}\frac{1}{f'(w)}+z\frac{dw}{dx}}{\left\{\frac{dv}{dy}+\frac{dz}{dy}f(w)\right\}\frac{1}{f'(w)}+z\frac{dw}{dy}}=0, \quad . \quad . \quad . \quad (21)$$

$$c+v+zf(w)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Of course

$$f'(w)\frac{dw}{dx}=\frac{d}{dx}f(w).$$

It is clear, therefore, that (21) is satisfied by the relation

$$w-h=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where h is any arbitrary constant if $f'(h)=$ infinity.

The differential equation (21) is, therefore, satisfied both by the complete primitive (22) and by the equation (23), which is a singular solution of (21), if it is not contained in (22) by giving a constant value to (c).

This form of the complete primitive, to which all primitives may be reduced, gives the characteristic singularity to the differential equation (21), which is derived from it

by ordinary differentiation, and the term *singular solution* of (21) can be justified only on the ground that it serves admirably to distinguish the two kinds of solution referred to in (22) and (23).

In the general theorems when a solution is said to be singular, it is meant to be so only when it is not contained in the complete primitive by giving a constant value to (c) independent of x. (See Boole's Diff. Eqs. p. 163, 2nd ed.)

8. The following examples will illustrate the general formulæ:—

Let

$$\left(\frac{dy}{dx}\right)^2 - 2\sqrt{x+y}\,\frac{dy}{dx} + 2\sqrt{x+y} = 1. \quad . \quad (24)$$

Required the singular solution, if any.

This equation admits of the form

$$\left(\frac{dy}{dx} - 2\sqrt{x+y} + 1\right)\left(\frac{dy}{dx} - 1\right) = 0. \quad . \quad . \quad (25)$$

The condition of equal roots is evidently

$$\sqrt{x+y} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

But this is not a solution of (24), therefore the principle of equal roots fails to give the singular solution.

Again,

$$\frac{dp}{dy} = \frac{1}{\sqrt{x+y}} = \text{infinity, if } x+y=0,$$

And this is the singular solution required.

The complete primitive of (24) is

$$(c-x)^2 - (y+\sqrt{x+y})(c-x) + y\sqrt{x+y} = 0. \quad . \quad (27)$$

The condition of equal roots is

$$x+y=y^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

which is not a solution of (24), therefore the principle of equal roots applied to the complete primitive fails also to give the singular solution.

The elimination of (c) from (27) by the condition $\frac{dy}{dc}=0$ and $\frac{dx}{dc}=0$ leads to the equation (28). The usual methods, therefore, of finding the singular solution from the complete primitive fail in this example.

The primitive (27) admits of the form

$$(c+\sqrt{x+y}-x)(c+y-x)=0,$$

from which we have the two equations

$$c+\sqrt{x+y}=x,$$
$$c+y=x.$$

The first represents a series of parabolic curves, and the second a series of straight lines to which the singular solution $x+y=0$ is perpendicular.

Let

$$\frac{dy}{dx}=\frac{y}{x}+\frac{y^2}{x\sin\frac{x}{y}}. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Required the singular solution if there is one.

This example is given by Sir James Cockle, F.R.S. &c. (See 'Quarterly Journal of Mathematics,' No. 54, 1876.)

Its complete primitive is

$$c+x-\cos\frac{x}{y}=0. \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Compare this equation with (22), then $v=x$, $z=-1$, and $f(w)=\cos\frac{x}{y}$. Therefore $f'(w)=-\sin\frac{x}{y}$, which does not become infinite for any value of x or y.

The equation $x=0$ satisfies (29), but it is a particular

solution, as may be inferred from (30) by expanding $\cos.\frac{x}{y}$ and taking $c=1$.

The condition $x=0$ gives $\frac{dp}{dy}=$ infinity and $\frac{d}{dx}\left(\frac{1}{p}\right)=0$.

Boole says, "If $\frac{dp}{dy}=$ infinity leads to a solution which does not involve (y) in its expression, nothing is to be inferred whether it is singular or not. Then the proper test is $\frac{d}{dx}\left(\frac{1}{p}\right)=$ infinity." Why not? the inference here is that $x=0$ is a particular solution of (29), which has been already proved.

The equation $y=0$ does not satisfy (29), as may be seen from the following development:—

Since

$$\frac{y^2}{x\sin\frac{x}{y}}=\frac{y^2}{x}\operatorname{cosec}\frac{x}{y},$$

$$=\frac{y^2}{x}\left\{\frac{y}{x}+\frac{x}{6y}+\frac{7x^3}{360y^3}+\&c.,\right\}$$

$$=\frac{y^3}{x^2}+\frac{y}{6}+\frac{7x^2}{360y}+\frac{31x^4}{15120y^3}+\&c.$$

$$\therefore\ \frac{0}{x\sin\infty}=\text{infinity}.$$

Therefore $\frac{dy}{dx}=$ infinity in (29) when $y=0$.

9. The following particular cases of (21) and (22) are interesting.

Let

$$f(w)=\log w,\qquad \therefore\ f'(w)=\frac{1}{w}.$$

Equations (21) and (22) become by substitution

$$\frac{dy}{dx}+\frac{\left\{\frac{dv}{dx}+\frac{dz}{dx}\log w\right\}w+z\frac{dw}{dx}}{\left\{\frac{dv}{dy}+\frac{dz}{dy}\log w\right\}w+z\frac{dw}{dy}}=0, \quad . \quad . \quad (31)$$

$$c+v+z\log w=0. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Now (31) is satisfied by $w=0$, which, however, is not a singular, but a particular solution by taking (c) equal to infinity.

From (32)

$$\log w=-\frac{c+v}{z},$$

$$\therefore\ w=\epsilon^{-\frac{c+v}{z}}=\frac{1}{\epsilon^{\frac{c}{z}}\epsilon^{\frac{v}{z}}}.$$

$$\therefore\ w=0, \text{ when } c=\text{infinity}.$$

Let

$$f(w)=w^{\frac{1}{n}}, \text{ where } (n) \text{ is an integer.}$$

$$\therefore\ f'(w)=\frac{1}{nw^{1-\frac{1}{n}}}.$$

Substitute these values in (21) and (22), then

$$\frac{dy}{dx}+\frac{n\left(\frac{dv}{dx}+w^{\frac{1}{n}}\frac{dz}{dx}\right)w^{1-\frac{1}{n}}+z\frac{dw}{dx}}{n\left(\frac{dv}{dy}+w^{\frac{1}{n}}\frac{dz}{dy}\right)w^{1-\frac{1}{n}}+z\frac{dw}{dy}}=0, \quad . \quad . \quad (33)$$

$$c+v+zw^{\frac{1}{n}}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

This equation, viz. (33), is satisfied by $w=0$, which is a singular solution, if it is not contained in the complete primitive (34) by giving a constant value to (c).

Again, let

$$f(w)=\cos w, \text{ then } f'(w) = -\sin w;$$

substitute these values in (21) and (22), then

$$\frac{dy}{dx}+\frac{\left(\frac{dv}{dx}+\frac{dz}{dx}\cos w\right)\frac{1}{\sin w}-z\frac{dw}{dx}}{\left(\frac{dv}{dy}+\frac{dz}{dy}\cos w\right)\frac{1}{\sin w}-z\frac{dw}{dy}}=0, \quad . \quad . \quad (35)$$

$$c+v+z\cos w=0. \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Now $w=0$ is not a solution of (35), since $\sin w$ is zero, and no value which can be given to (w) will satisfy the equation $\sin w=$infinity.

Let

$$\frac{dy}{dx}=\frac{y\log y}{x}, \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$\therefore\ c-\frac{\log y}{x}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

is its complete primitive. This example is from Boole's Diff. Eqs. Supp. vol. p. 17.

In equation (34) put $v=0$, $z=-\log y$, $n=-1$, and $w=x$; then $x=0$ is a solution which satisfies (37), but which is not contained in (38) by giving a constant value to (c), therefore it is a singular solution of (37). Boole rejects the solution $x=0$ because it does contain in its expression (y). For what reason this rejection is made does not appear.

By reference to equations (31) and (32), put $v=0$, $z=-\frac{1}{x}$, and $w=y$, then $y=1$ is a solution of (37), but it is a particular solution by giving $c=0$ in the primitive $y=\epsilon^{cx}$.

Boole has obtained the solution $y=0$ in two places, viz.

Diff. Eqs. p. 161, 2nd ed., and supp. vol. p. 17, by means of the condition $\frac{dp}{dy}$=infinity, and thereby plunged himself into a complete mathematical labyrinth, from which his great genius would not enable him to extricate himself. He states that "the value of (c) is not wholly independent of (x)," and "regards $y=0$ as a singular solution;" whereas $y=0$ is a particular solution, where $c=-$infinity. For the value of (c) to be dependent, in any degree, upon the value of (x) is in direct opposition to the spirit of the definition of a singular solution.

By paragraph (9) it appears that no differential equation, whose primitive is of the form $c+v+z\log w=0$, can have a singular solution.

10. There are three questions which demand special consideration in the subject of singular solutions of differential equations, viz.—

1. The singular solution as derived from the complete primitive.
2. The singular solution as derived from the differential equation.
3. Given a solution of a differential equation, is it a singular or a particular integral?

These questions have to some extent been replied to in paragraphs 1 to 5.

11. With respect to the first question, it may be observed that every complete primitive can or cannot be put into the form (22). When the former is possible then the singular solution is obvious from the primitive itself, but if the latter prevails then the conclusion is that there is no singular solution.

It may be noticed here that other terms of the form

$z_1 f_1(w_1)$ may be added to the primitive (22), thereby giving rise to other singular solutions, $w_1 = h_1$ &c.

12. The second problem is more difficult to answer than the first, and seems to have been first generally considered by Laplace himself. (See Boole's Diff. Eqs. p. 174, 2nd ed.)

Let

$$\frac{dy}{dx}+r=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

be the differential equation whose singular solution is required, (r) being a function of x, y.

By reference to equation (21) $f'(w)$, being the special object of inquiry, is the factor which makes (21) an exact differential equation, therefore (39) admits of the form

$$\frac{dy}{dx}+\frac{f'(w)r}{f'(w)}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

If, therefore, (40) is an exact differential equation, then

$$\frac{d}{dx}f'(w)=\frac{d}{dy}\{f'(w)r\},$$

or

$$\frac{dp}{dy}=\frac{f''(w)}{f'(w)}\left\{r\frac{dw}{dy}-\frac{dw}{dx}\right\} \quad . \quad . \quad . \quad . \quad (41)$$

Since

$$p=-r \text{ and } \frac{dp}{dy}=-\frac{dr}{dy}.$$

Now by paragraph (7) the singular solution $w-h=0$ makes $f'(h)=$ infinity, therefore $f'(w)$ must, for a singular solution, necessarily be of the form

$$f'(w)=\frac{\phi(w)}{w-h}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

whereof the function $\phi(w)$ does not vanish for $w=h$.

Differentiate (42) with respect to (w), then

$$f''(w)=\frac{\phi'(w)}{w-h}-\frac{\phi(w)}{(w-h)^2};$$

substitute the values of $f'(w)$ and $f''(w)$ in (41), then

$$\frac{dp}{dy}=\left(\frac{\phi'(w)}{\phi(w)}+\frac{1}{h-w}\right)\left(r\frac{dw}{dy}-\frac{dw}{dx}\right). \quad . \quad . \quad (43)$$

When, however, $w=h$, which is the case for a singular solution, then the equation (43) becomes infinite on the dexter side; hence $\frac{dp}{dy}=$infinity, providing $\left(r\frac{dw}{dy}-\frac{dw}{dx}\right)$ does not vanish.

The condition, therefore, which satisfies $\frac{dp}{dy}=$infinity, and also satisfies the differential equation (39), may be a singular solution, and will be if it is not contained in the primitive by giving a constant value to (c).

It is not difficult to show that the complete primitive of (39), in terms of its integrating factor $f'(w)$, is

$$c+\int f'(w)\,dy+\int rf'(w)\,dx-\iint\frac{df'(w)}{dx}\,dydx=0. \quad . \quad (43')$$

13. The solution of the third question is more difficult still than either the first or second. It will be seen by what follows that Boole's solution to this question is not regarded as altogether satisfactory (see supp. vol. pp. 28, 29, 30).

The assertion by Boole that $F(x, 0)$ is a constant, or rather $F(x, 0)$ is not a function of x, for a particular solution is not universally correct. The theorem, therefore, in problem (10), p. 28, supp. vol., is faulty in proof; it is also difficult in its application and uncertain in its utterance.

Observe the following example:—

Let

$$\left(\frac{dy}{dx}\right)^2-(x+y+1)\frac{dy}{dx}-2x(x-y-1)=0, \quad . \quad (44)$$

whose complete primitive is

$$c^2-c\{x+x^2-y-\log(x-y)\}+(x^2-y)\{x-\log(x-y)\}=0. \quad . \quad (45)$$

Now $x-y=0$ is a solution of (44); is it a singular or a particular solution? The primitive can be put in the form

$$\{c-x+\log(x-y)\}\{c+y-x^2\}=0. \quad . \quad . \quad (46)$$

Therefore

$$c-x+\log(x-y)=0,$$

from which

$$x-y=\frac{\epsilon^x}{\epsilon^c}=0, \text{ when } c=\text{infinity}.$$

Hence $x-y=0$ is a particular solution, the constant (c) being infinite. It remains to eliminate (y) from (44) and (45) by means of

$$w=x-y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Then

$$\left(\frac{du}{dx}\right)^2+(2x-w-1)\frac{dw}{dx}-w(2x-1)=0, \quad . \quad (48)$$

$$c^2-c\{x^2+w-\log w\}+(x^2-x+w)(x-\log w)=0. \quad . \quad (49)$$

Now $w=0$ is a particular solution, and satisfies (48), but the primitive (49) still remains a function of x, which is in opposition to the principle used by Boole (supp. vol. p. 28).

14. Boole's proof of a theorem from Euler.

First eliminate (y) from the given differential equation, and its complete primitive by the relation $w=\phi(x, y)$; then

$$\frac{dw}{dx}=\psi(x, w), \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

whose complete primitive is

$$c+\mathrm{F}(x, w)=0, \quad . \quad . \quad . \quad . \quad . \quad (51)$$

$$\therefore \int_0^w \frac{dw}{\psi(x, w)}=0 \text{ or } \infty \quad . \quad . \quad . \quad (52)$$

makes $w=0$ a singular or a particular solution.

Differentiate (51) with respect to x, then

$$\frac{dw}{dx}=-\frac{\frac{d}{dx}\mathrm{E}(x, w)}{\frac{d}{dw}\mathrm{F}(x, w)}.$$

Hence

$$\psi(x, w)=-\frac{\frac{d}{dx}\mathrm{E}(x, w)}{\frac{d}{dw}\mathrm{F}(x, w)},$$

from which there results

$$\int_0^w \frac{dw}{\psi(x, w)}=-\int_0^w \frac{\frac{d}{dw}\mathrm{F}(x, w)}{\frac{d}{dx}\mathrm{F}(x, w)}dw. \quad . \quad . \quad (53)$$

The integral on the dexter side, as given by Boole, is equal to $\mathrm{H}.\mathrm{F}(x, w)$, "where (H) is some value intermediate between the greatest and least values of 1 upon $\frac{d}{dx}\mathrm{F}(x, w)$ within the limits of integration." See supp. vol. p. 29.

This integral, however, is regarded as unsatisfactory, therefore the following is proposed in its place, viz.

$$\int \frac{dw}{\psi(x, w)}=-\frac{w\frac{d}{dw}\mathrm{F}(x, \beta w)}{\frac{dx}{d}\mathrm{F}(x, \beta w)}, \quad . \quad . \quad . \quad (54)$$

where (β) is a proper fraction (see Todhunter's Diff. Calculus, p. 81, third ed.). Nothing as yet has been said with respect to the conditional equation $w=\phi(x, y)$ by means of which (50) and (51) have been obtained. The above general integral obtains whether $w=0$ is á solution of the differential equation or otherwise. Between the limits zero and (w) the integral becomes

$$\int_0^w \frac{dw}{\psi(x, w)} = -\frac{w\frac{d}{dw}F(x, w)}{\frac{d}{dx}F(x, w)}. \quad . \quad . \quad . \quad (55)$$

From paragraph (13) it appears that $F(x, 0)$ may be and frequently is a function of x when $w=0$ is a particular solution. Boole's argument in supp. vol. p. 28, is not universally correct.

The inferences from (55) are as follows:—

When $w=0$ and $F(x, 0)$ is a function of x, then

$$\int_0^w \frac{dw}{\psi(x, w)} = 0. \quad . \quad . \quad . \quad . \quad . \quad (56)$$

But whether $w=0$ is a singular or a particular solution the equation (56) does not determine.

When $w=0$ and $F(x, 0)$ is a constant, then

$$\int_0^w \frac{dw}{\psi(x, w)} = \frac{0}{0}. \quad . \quad . \quad . \quad . \quad . \quad (57)$$

This equation, which is indeterminate, gives $w=0$ as a particular solution if $\frac{d}{dw}F(x, 0)$ is finite. Boole's example, supp. vol. p. 31, confirms the truth of (57) as log $\left(\frac{\log y}{\log 0}\right)$, is indeterminate when $y=0$. There is nothing whatever to show, in Boole's demonstration, that $w=0$ is a singular solution of the envelope species.

15. To find, without the aid of the complete primitive, whether a solution $w=0$ of a differential equation of the first order and of the nth degree is singular or particular is a problem the answer to which is not by any means obvious, and, if it be solved, which is doubtful, by the process given by Boole (see supp. vol. p. 30), the solution is accomplished by a troublesome transformation, accompanied by a definite integral, which, by the bye, is often impracticable.

The following answer to this difficult problem may be interesting.

Reduce, by algebraic methods, the given differential equation into its quartic factors, of which the following is one, viz.

$$\left(\frac{dy}{dx}+r\right)\left(\frac{dy}{dx}+s\right)=0, \quad . \quad . \quad . \quad . \quad (58)$$

where r, s are given functions of x, y.

If the solution $w=0$ be of the envelope species the results of the substitution of (w) in (r) and (s) will be equal.

It will now be necessary to compare (58) with the first factor in equation (11), paragraph 4, omitting the affixes; then

$$2\sqrt{w}\frac{dv}{dx}-\frac{dw}{dx}=r\left(2\sqrt{w}\frac{dv}{dy}-\frac{dw}{dy}\right), \quad . \quad . \quad . \quad (59)$$

$$2\sqrt{w}\frac{dv}{dx}-\frac{dw}{dx}=s\left(2\sqrt{w}\frac{dv}{dy}-\frac{dw}{dy}\right); \quad . \quad . \quad . \quad (60)$$

from these equations

$$(s-r)\frac{dv}{dx}=(r+s)\frac{d\sqrt{w}}{dx}-2rs\frac{d\sqrt{w}}{dy}, \quad . \quad . \quad (61)$$

$$(s-r)\frac{dv}{dy}=2\frac{d\sqrt{w}}{dx}-(r+s)\frac{d\sqrt{w}}{dy}. \quad . \quad . \quad . \quad (62)$$

From these equations (v), the only unknown element in the primitive $c+v\pm\sqrt{w}=0$, when $\sqrt{w}$ is given, can be easily found by the usual process, as given by Boole (see Diff. Eqs. pp. 48, 49, 2nd ed.). If, however, the solution $w=0$ has unfortunately dropped a factor, it will be a serious matter, in many cases, to pick it up, as $wz^2=0$ must be used in (61) and (62) instead of $w=0$, where z is an unknown quantity to be determined.

It is therefore obvious from the primitive that the solution $w=0$ cannot be particular if (v) contains x or y, except in the case of (v) being a function of (w).

If the dexter side of (61) and (62) are functions of x, y, the solution $w=0$ is singular and of the envelope species.

Example:—

$$xp^2-2yp+4x=0 \text{ (from Glaisher)}, \quad . \quad . \quad (63)$$

$$\therefore\ p-\frac{y}{x}\pm\frac{1}{x}\sqrt{y^2-4x^2}=0.$$

Compare this example with (58), (61), (62); then

$$r+s=-\frac{2y}{x}, \quad rs=4, \quad s-r=\frac{2}{x}\sqrt{y^2-4x^2}.$$

The condition of equal roots of (63) is $y=\pm 2x$, which is a solution of (63); then

$$w=y^2-4x^2,$$

$$\frac{dw}{dx}=-8x,$$

$$\frac{dw}{dy}=2y.$$

Substitute these values in (61) and (62), then

$$\frac{dv}{dx}=0, \quad \frac{dv}{dy}=1, \quad \text{therefore } v=y.$$

Hence $w=0$ is a singular solution of (63) of the envelope species.

16. By reference to equation (11) and by dropping the affixes to the variables (v) and (w) it is easy to obtain

$$\left\{4w\left(\frac{dv}{dy}\right)^2-\left(\frac{dw}{dy}\right)^2\right\}p^2+2p\left\{4w\frac{dv}{dx}\frac{dv}{dy}-\frac{dw}{dx}\frac{dw}{dy}\right\}$$

$$+4w\left(\frac{dv}{dx}\right)^2-\left(\frac{dw}{dx}\right)^2=0. \quad . \quad . \quad . \quad (64)$$

This equation, which is of the second degree in (p), is expressed in terms of its primitive $c+v\pm\sqrt{w}=0$, and always has a singular solution $w=0$ when (v) is not a constant.

A differential equation of the second degree and the first order, which is not of the form (64), cannot have a singular solution.

Professor Cayley says (see 'Messenger of Mathematics,' vol. vi. p. 23) that

$$\mathrm{L}p^2+2\mathrm{M}p+\mathrm{N}=0 \quad . \quad . \quad . \quad . \quad . \quad (65)$$

has not in general any singular solution when L, M, N are rational and integral functions of x, y.

If I apprehend correctly the meaning of this statement, it seems difficult to reconcile it with equation (64), the terms of which can always be made rational and integral functions of x, y by proper assumptions of (v) and (w).

Now, equation (65) admits of the form

$$\mathrm{L}^2p^2+2\mathrm{ML}p+\mathrm{M}^2-\mathrm{N}=0. \quad . \quad . \quad . \quad (66)$$

The condition of equal roots is $\mathrm{N}=0$, and this equation is a solution of (66) if

$$\mathrm{M}\frac{d\mathrm{N}}{dy}=\mathrm{L}\frac{d\mathrm{N}}{dx}. \quad . \quad . \quad . \quad . \quad . \quad (67)$$

Substitute this value in (66) and it becomes

$$L\frac{dN}{dy}p+L\frac{dN}{dx}\mp\sqrt{N}=0. \quad . \quad . \quad . \quad (68)$$

This equation, which is of the second degree in (p), is expressed in terms of the differential equation (66), and has in general a singular solution $N=0$ of the envelope species.

Equation (68) includes all the particular examples given by Professor Cayley and Mr. Glaisher in the papers already referred to.

17. Professor Cayley has considered the particular case

$$Lp^2-N=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

The condition of equal roots of (69) is $N=0$, which is a solution of (69) if N is a function of y only.

The equation $L=0$ is a solution of (69) if L is a function of x only. Hence, examples 3, 4 given by Professor Cayley can have no singular solutions. Their complete primitives are respectively

$$c+2x\pm(\sin^{-1}y+y\sqrt{1-y^2})=0, \quad . \quad . \quad (70)$$

$$c+\sin^{-1}y+y\sqrt{1-y^2}\pm(\sin^{-1}x+x\sqrt{1-x^2})=0, \quad (71)$$

which are the only solutions of these two examples.

Compare (69) with (57), then $r+s=0$ and $s=\sqrt{\frac{N}{L}}$,

$$\frac{dv}{dx}=\sqrt{\frac{N}{L}}\frac{d\sqrt{w}}{dy}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

$$\frac{dv}{dy}=\sqrt{\frac{L}{N}}\frac{d\sqrt{w}}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

The conditional equation is

$$\frac{d}{dy}\left\{\sqrt{\frac{\mathrm{N}}{\mathrm{L}}}\frac{d\sqrt{w}}{dy}\right\}=\frac{d}{dx}\left\{\sqrt{\frac{\mathrm{L}}{\mathrm{N}}}\frac{d\sqrt{w}}{dx}\right\}. \quad . \quad (74)$$

Equations (72), (73) determine the nature of the solutions $\mathrm{N}=0$ and $\mathrm{L}=0$.

This equation must be satisfied by a proper value of $\sqrt{w}$, or otherwise equations (72) and (73) cannot be integrated.

Example:—

$$p^2=1-y^2 \text{ (from Cayley)}. \quad . \quad . \quad . \quad . \quad (75)$$

The condition of equal roots is $y=\pm 1$, which is a solution of (75), but whether it is singular or particular depends upon (72) and (73). In this case $s+r=0$, $\mathrm{L}=1$, $s=\sqrt{1-y^2}$.

The equation (74) is satisfied by

$$\sqrt{w}=\sin x\sqrt{1-y^2}. \quad . \quad . \quad . \quad . \quad . \quad (76)$$

Then (72) and (73) become

$$\frac{dv}{dx}=-y\sin x,$$

$$\frac{dv}{dy}=\cos x,$$

from which

$$v=y\cos x,$$

therefore the complete primitive is

$$c+y\cos x\pm\sin x\sqrt{1-y^2}=0,$$

and $y=\pm 1$ is a singular solution of the envelope species.

Example:—

$$p^2(1-x^2)=1-y^2 \text{ (from Cayley)}. \quad . \quad . \quad . \quad (77)$$

The condition of equal roots is in this case $x=\pm 1$ and

$y=\pm 1$, which satisfy (77), but equations (72) and (73) will affirm the nature of these solutions. Put $L=1-x^2$, $N=1-y^2$; then the conditional equation (74) is satisfied by $\sqrt{w}=\sqrt{1-x^2 \, . \, 1-y^2}$, and (72), (73) become

$$\frac{dv}{dx}=-y,$$

$$\frac{dv}{dy}=-x.$$

From these two equations

$$v=-xy,$$

therefore the complete primitive is

$$c+xy\pm\sqrt{1-x^2 \, . \, 1-y^2}=0. \quad . \quad . \quad . \quad (78)$$

Hence $y=\pm 1$ and $x=\pm 1$ are singular solutions of the envelope species. Equation (78) admits of the form

$$c^2+2cxy+y^2+x^2-1=0, \quad . \quad . \quad . \quad (79).$$

from which

$$y=-cx\pm\sqrt{1-c^2}\sqrt{1-x^2}. \quad . \quad . \quad . \quad (80)$$

It is seen from (80) that neither (c) nor (x) can exceed unity, for if they do (y) becomes impossible.

Because (c^2-1) is negative the curves (79) are a system of ellipses (see Todhunter's 'Conics,' third ed. p. 239), whose principal axes and centre are the diagonals, and their intersection, of the square $y=\pm 1$, $x=\pm 1$.

The conditional equation (74) is satisfied also by $\sqrt{w}=y\sqrt{1-x^2}$ and $\sqrt{w}=x\sqrt{1-y^2}$, giving the primitives

$$\left.\begin{array}{l} c+x\sqrt{1-y^2}\pm y\sqrt{1-x^2}=0, \\ c+y\sqrt{1-x^2}\pm x\sqrt{1-y^2}=0. \end{array}\right\} \quad . \quad . \quad . \quad (81)$$

Mr. Glaisher states the condition of equal roots of (77) to be $1-x^2 . 1-y^2=0$. All I can say at present is that $1-x^2=0$ satisfies the condition of equal roots in a peculiar way.

It may be stated that (77) can be replaced by its equivalent

$$(1-y^2)\left(\frac{dx}{dy}\right)^2=1-x^2. \quad . \quad . \quad . \quad . \quad (82)$$

The roots of which, with respect to $\left(\frac{dx}{dy}\right)$, are equal when $x=\pm 1$, a condition which satisfies (82). This principle can be applied to equation (65), with a view to show the cases in which $L=0$ may produce a singular solution as well as $N=0$.

18. Professor Cayley states that if $Lp^2+2Mp+N=0$ has a singular solution, it is either $M^2-LN=0$ or a factor of M^2-LN equal zero. To this proposition may be added another, viz.

$$Lp^2+2Mp-L\frac{dM}{dx}:\frac{dM}{dy}=0$$

has a singular solution $M=0$, which does not appear to be included by the above proposition of Professor Cayley's.

XVIII. *On the Formula for the Intensity of Light transmitted through an Absorbing Medium as deduced from Experiment.* By James Bottomley, B.A., D.Sc., F.C.S.

Read October 17th, 1882.

In a former communication an experimental method was suggested for testing the validity of an assumed law of intensity of light that has passed through an absorbing medium. The method was this: take two surfaces of different degrees of brightness, survey them through some absorbing medium, adjust the lengths of the columns so that the intensities shall be the same; then, if the law of absorption be true, the intensities will again be equal if both columns are increased by the same length. Some experiments which I made gave results in agreement with the theory. In these experiments surfaces of different degrees of whiteness were observed through a grey solution. The error arising from the finite extent of the surface is small, and the mean intensity which we observe may be taken as the intensity of the central ray.

But suppose we had started with no hypothesis as to the form of the function expressing the intensity of transmitted light, but had found, as an experimental result, that when the intensities are equal, they remain equal, when the columns receive equal increments; what form for the function might be deduced from such a result?

Suppose there are two lights of initial brightness, I_0 and I'_0 respectively, then if x and y be the lengths of the

absorbing columns for the transmitted light in one case, we shall have

$$I_x = I_0 \phi(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and in the other case

$$I_y = I'_0 \phi(y), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

ϕ being the unknown function which it is required to determine. Since these two intensities are equal,

$$I_0 \phi(x) = I'_0 \phi(y). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If both columns receive an equal increment, the intensities will again be equal. Let this common increment be denoted by κ, then

$$I_0 \phi(x+\kappa) = I'_0 \phi(y+\kappa). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

These equations will still hold if $y=0$.

In this case the duller surface is observed directly, and the length of the absorbing column over the brighter surface is increased until they are brought to the same intensity. Since when $y=0$, $I'_y = I'_0$, therefore $\phi(0)=1$.

Equations (3) and (4) will become

$$I_0 \phi(x) = I'_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$I_0 \phi(x+\kappa) = I'_0 \phi(\kappa). \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Expand (6) in terms of κ,

$$I_0 \left\{ \phi(x) + \frac{d\phi(x)}{dx}\kappa + \frac{d^2\phi(x)}{2dx^2}\kappa^2 + \&c. \right\}$$
$$= I'_0 \left(1 + \phi'(0)\kappa + \frac{\phi''(0)}{2}\kappa^2 + \&c. \right).$$

Since the first terms of the expansion on each side are equal, the equation may be written

$$I_0 \left\{ \frac{d\phi x}{dx}\kappa + \frac{d^2\phi x}{2dx^2}\kappa^2 + \&c. \right\} = I'_0 \left(\phi'(0)\kappa + \frac{\phi''(0)}{2}\kappa^2 + \&c. \right).$$

Dividing both sides by κ and diminishing κ without limit, the resulting equation will be

$$I_0 \frac{d\phi(x)}{dx} = I'_0 \phi'(0) ;$$

eliminating I_0 and I'_0 by (5),

$$\frac{d\phi(x)}{dx} = \phi'(0)\phi(x).$$

The integral of this is

$$\log \phi(x) = \phi'(0)x + c,$$

or

$$\phi(x) = Ce^{\phi'(0)x}.$$

When $x = 0$,

$$\phi(x) = 1.$$

Therefore

$$C = 1.$$

Also as x increases, the intensity diminishes, therefore $\phi'(0)$ must be some negative constant; let it be denoted by $-m$. Then the equation becomes

$$\phi(x) = e^{-mx},$$

and equation (1) may be written

$$I_x = I_0 \epsilon^{-mx}.$$

Hence experiment leads to the same form for the function as the hypothetical form with which we started.

If, in the above investigation, we had made the length of the column invariable, and x denoted a mass of some colouring-matter which undergoes no decomposition on dilution, we might have obtained experimentally the form of the function expressing the intensity of the light transmitted through a column of fluid of invariable length, containing a variable quantity of colouring-matter.

In the above remarks I have supposed we are dealing with homogeneous light, or with white light which has penetrated a medium containing soluble black in solution. To apply the formulæ generally we must prefix to them the sign of summation.

In seeking *à priori* the law of transmitted light, we might have reasoned as follows, which involves less assumption than Herschel's reasoning.

Suppose we have a column of any length; conceive it divided anywhere into two lengths x and y by an imaginary plane. Let I_0 be the initial intensity of light; after penetrating the column x we shall have

$$I_x = I_0 \phi(x).$$

But if light of intensity I_x penetrate a column of length y, the transmitted light will be $I_x\phi(y)$, or by substitution $I_0\phi(x)\phi(y)$. Since the length of the column is $x+y$, the emergent light will also be expressed by $I_0\phi(x+y)$. Equating these two expressions for the same quantity, there results

$$\phi(x)\phi(y) = \phi(x+y).$$

It is well known that this functional equation is satisfied by an exponential form.

XIX. *On the Intensity of Light that has been transmitted through an Absorbing Medium, in which the Density of the Colouring-matter is a Function of the Distance traversed.* By JAMES BOTTOMLEY, B.A., D.Sc., F.C.S.

Read October 17th, 1882.

IN previous papers it has been supposed that the absorbing matter was uniformly distributed throughout the medium. For such a case the laws $I = \Sigma a\epsilon^{-mt}$ and $I = \Sigma a\epsilon^{-\mu q}$ are applicable, t denoting the length of the absorbing column, and q the mass of colouring-matter contained in it. In the present paper these laws are applied to the case of light passing through media of variable density. Such cases occur in nature; for instance, the atmosphere is of variable density, increasing as we approach the earth; also we might have coloured glass in which the colouring-matter is not uniformly distributed, or the case of a coloured soluble salt on which water is poured; the colour in the immediate vicinity of the salt is most intense, and gradually fades as the distance increases. The same reasoning will also apply very approximately to fluids containing in suspension very finely divided matter in layers of variable density, or to an atmosphere charged with very fine dust.

For simplicity I shall suppose that we are dealing with homogeneous light, or with white light that has passed through a grey solution.

Suppose that a ray has penetrated a length t of a variable medium, and that the intensity is I when it falls

on a surface for which the coefficient of transmission is ϵ^{-m}. Let us take a plate of the medium of thickness Δt, and let $\epsilon^{-(m+\Delta m)}$ be the coefficient of transmission at the upper surface. If I′ be the transmitted light,

$$I' < I\epsilon^{-m\Delta t},$$

$$> I\epsilon^{-(m+\Delta m)\Delta t}.$$

Expanding, and putting ΔI for $I'-I$, this becomes

$$\frac{\Delta I}{I} < -m\Delta t + m^2\frac{\Delta t^2}{2} - \&c.,$$

$$> -(m+\Delta m)\Delta t + (m+\Delta m)^2\frac{\Delta t^2}{2} - \&c.$$

Ultimately we get

$$d\log I = -m dt. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now suppose m to be some function of t; if this be some integral form, then we may determine I.

If light has passed through a homogeneous medium t units long, then $I = I_0\epsilon^{-mt}$. If the length be unity, $I = I_0\epsilon^{-m}$. If the unit length contain q units of colouring-matter, we also have $I = I_0\epsilon^{-\mu q}$. Equating these two values of I, there results the equation

$$m = \mu q.$$

Let d be the density of the colouring-matter, then d will be proportional to q. If the unit of density be that due to the distribution of the unit mass through unit volume, then we may replace q by d. Now suppose the density variable and some function of t, so that we may write

$$d = \phi(t).$$

Substituting in equation (1) we get

$$d\log I = -\mu\phi(t)\,dt. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let $\chi(t) = \int \phi(t)\,dt$. Then, integrating equation (2), we obtain

$$\log \mathrm{I} = -\mu\chi(t) + c,$$

which may also be written in the form

$$\mathrm{I} = \mathrm{C}\epsilon^{-\mu\chi(t)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

To determine the constant we must know the initial values of I and t. Suppose that when $t = 0$, $\mathrm{I} = \mathrm{I}_0$; then

$$\mathrm{I}_0 = \mathrm{C}\epsilon^{-\mu\chi(0)},$$

and substituting for C in (3) we get

$$\mathrm{I} = \mathrm{I}_0\epsilon^{\mu\chi(0)}\epsilon^{-\mu\chi(t)}.$$

This will be the general expression for the intensity of light which has penetrated a length t of a medium of variable density.

I. Suppose the density to vary as the distance from the plane of incidence; then we have

$$\phi(t) = nt,$$

where n is some constant;

$$\chi(t) = \frac{nt^2}{2}$$

and

$$\mathrm{I} = \mathrm{C}\epsilon^{-\mu n\frac{t^2}{2}}.$$

If I_0 be the intensity when $t = 0$, we get

$$\mathrm{I}_0 = \mathrm{C}.$$

This determines the constant. If for nt we substitute d, we obtain the following relationship between the intensity and the density at any point:—

$$\mathrm{I} = \mathrm{I}_0\epsilon^{-\frac{\mu}{2n}d^2}.$$

II. As another example, suppose the absorbing medium to be an elastic fluid arranged in concentric layers about an attracting sphere of radius R, the law of attraction being that of the inverse square. Let t be the distance of any point from the centre, then if f be the attractive force,

$$f=\frac{m}{t^2},$$

m being a constant. It may be shown that the density at any point will be given by the equation

$$d=\mathrm{A}\epsilon^{\frac{m}{at}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where a is a constant denoting the relationship of the density to the pressure, and A another constant which may be determined as follows:—

Let D be the density at the surface of the sphere; then making $t=\mathrm{R}$, we get

$$\mathrm{A}=\mathrm{D}\epsilon^{-\frac{m}{a\mathrm{R}}}.$$

Hence we have

$$\phi(t)=\mathrm{D}\epsilon^{-\frac{m}{a\mathrm{R}}}\epsilon^{\frac{m}{at}},$$

and for the intensity of the transmitted light we have

$$\mathrm{I}=\mathrm{C}\epsilon^{-\mu\mathrm{D}\epsilon^{-\frac{m}{a\mathrm{R}}}\int\epsilon^{\frac{m}{at}}dt} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The value of C may be obtained by substituting for I its value at the surface of the sphere, and after integrating the quantity under the integral sign, substituting R for t.

Equation (4) may be written in the form

$$d=\mathrm{D}\epsilon^{-\frac{m}{a\mathrm{R}}}\epsilon^{\frac{m}{a(\mathrm{R}+\tau)}}.$$

If τ be small compared with R, we may write

$$d = \mathrm{D}\epsilon^{-\frac{m}{a\mathrm{R}}}\epsilon^{\frac{m}{a\mathrm{R}}\left(1-\frac{\tau}{\mathrm{R}}\right)}.$$

Let the attracting body be the earth; then for m we may substitute $g\mathrm{R}^2$, and we obtain

$$d = \mathrm{D}\epsilon^{-\frac{g}{a}\tau}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In this case the expression for the intensity becomes

$$\mathrm{I} = \mathrm{C}\epsilon^{\frac{a}{g}\mu\mathrm{D}\epsilon^{-\frac{g}{a}\tau}}.$$

To determine C make $\mathrm{I} = \mathrm{I}_0$ and $\tau = 0$. Then

$$\mathrm{I}_0 = \mathrm{C}^{\mu\frac{a}{g}\mathrm{D}},$$

and the expression for the intensity becomes

$$\mathrm{I} = \mathrm{I}_0\epsilon^{\mathrm{D}\mu\frac{a}{g}(\epsilon^{-\frac{g}{a}\tau}-1)}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This is the expression for the intensity of light which has passed through a length τ of the atmosphere, supposed to be of uniform temperature, neglecting the differences in the force of gravity at different heights. If the law of decrease of temperature with altitude be given, then a in (6) must be multiplied by some function of τ. When τ becomes large the expression (7) tends towards a limiting value

$$\mathrm{I} = \mathrm{I}_0\epsilon^{-\frac{a}{g}\mu\mathrm{D}}.$$

The atmosphere, especially in the lower layers, is never free from dust; of the whole atmospheric absorption a considerable portion would probably be due to this.

The relationship of the density at any point to the pressure is given by the equation

$$I = I_0 \epsilon^{-D\mu\frac{a}{g}\epsilon^{\mu\frac{a}{g}d}}.$$

In the previous cases the ray has been supposed to proceed through the atmosphere from the earth; if the path of the ray is through the atmosphere towards the earth, the formulæ will require a little alteration.

Let h be the distance of the luminous object from the earth, and I_0 its brightness at that distance, θ its distance from any layer of the atmosphere below it, and τ the distance of that layer from the earth. Then we have

$$h = \tau + \theta.$$

Hence the expression for the density becomes

$$d = D\epsilon^{-\frac{g}{a}(h-\theta)},$$

and the formula for the intensity of the transmitted light becomes

$$I = I_0 \epsilon^{\mu D\frac{a}{g}\epsilon^{-\frac{g}{a}h}\left(1-\epsilon^{\frac{g}{a}\theta}\right)}.$$

If, in this formula, we make $\theta = h$, and then make h large, we have for the limiting value of the intensity

$$I = I_0 \epsilon^{-\mu D\frac{a}{g}}.$$

In equation (7) the light is supposed to proceed through the atmosphere vertically. When this is not the case the expression for the intensity becomes more complex. In treatises on geometrical optics may be found the following differential equation of the path of a ray of light which has passed through a medium, of which the refractive

index at any point is a function of the distance from a fixed point,

$$\frac{d\theta}{dr}=\frac{C}{r\sqrt{\mu^2r^2-C^2}},$$

From this equation we obtain

$$\frac{ds}{dr}=\frac{r\mu}{\sqrt{r^2\mu^2-C^2}}.$$

Let R be the earth's radius, and z the apparent zenith distance at the place of observation; then if the refringent power of a gas be constant, so that $\mu^2-1=\kappa d$, we obtain

$$\frac{ds}{dr}=\frac{r\sqrt{(1+\kappa d)}}{\sqrt{r^2(1+\kappa d)-R^2\sin^2 z}}.$$

This leads to the following differential equation for determining the intensity of the transmitted light at any point:—

$$\frac{d\log I}{dr}=-\mu D\epsilon^{-g\frac{R}{a}}\epsilon^{\frac{gR^2}{ar}}\frac{r\sqrt{1+\kappa D\epsilon^{-g\frac{R}{a}}\epsilon^{\frac{gR^2}{ar}}}}{\sqrt{r^2\left(1+\kappa D\epsilon^{-g\frac{R}{a}}\epsilon^{\frac{gR^2}{ar}}\right)-R^2\sin^2 z}}.$$

III. We might also have the absorbing matter distributed through the medium in such a manner that the same value for the density would recur. Suppose, for instance, that the colouring-matter is distributed in parallel layers, and that the law of density is given by the equation

$$d=m-n\sin t,$$

where t denotes the distance from some plane, and m and n are constants, m being greater than n, the initial density will be m, and the value will recur whenever $\sin t$ vanishes. The density will have a maximum value $m+n$ whenever t is of the form $2\nu\pi+3\frac{\pi}{2}$, where ν has any

positive integral value including 0, also the density will have a minimum value $m-n$ whenever t is of the form $2\nu\pi+\frac{\pi}{2}$, where ν has any positive integral value including 0. A plate of such a medium, cut off by two parallel planes perpendicular to the layers of colouring-matter, when viewed against the light, would present the appearance of light and dark bands gradually increasing and diminishing in intensity. In this case we have

$$\chi(t)=\int(m-n\sin t)dt,$$

$$=mt+n\cos t,$$

and the formula for the intensity becomes

$$\mathrm{I}=\mathrm{C}\epsilon^{-\mu(mt+n\cos t)}.$$

To determine C, make $t=0$ and $\mathrm{I}=\mathrm{I}_0$; then

$$\mathrm{I}_0=\mathrm{C}\epsilon^{-\mu n}.$$

Hence we have

$$\mathrm{I}=\mathrm{I}_0\epsilon^{\mu n}\epsilon^{-\mu(mt+n\cos t)}.$$

At successive layers of maximum density we shall obtain

$$\mathrm{I}=\mathrm{I}_0\epsilon^{\mu n}\epsilon^{-\mu m\left(2\nu\pi+3\frac{\pi}{2}\right)},$$

where for ν must be substituted 0, 1, 2, &c.

The above may be written more briefly

$$\mathrm{I}=\mathrm{I}_0\mathrm{C}\epsilon^{-p\nu}.$$

Hence on emergence from successive layers of maximum density the values of the intensity are in geometrical progression, the first term being $\mathrm{I}_0\mathrm{C}$ and the ratio ϵ^{-p}.

At successive layers of minimum density we get

$$I = I_0 \epsilon^{\mu n} \epsilon^{-\mu m \left(2\nu\pi + \frac{\pi}{2}\right)},$$

which may be more briefly written

$$I = I_0 C' \epsilon^{-p\nu},$$

where for ν must be substituted in succession 0, 1, 2, &c. Hence the values of the intensity on emergence from layers of minimum density constitute another geometrical series, of which the first term is $I_0 C'$ and the common ratio ϵ^{-p}.

The nature of the absorption may be more clearly understood by representing the variables by coordinates.

Let ordinates represent intensities and abscissæ the distances traversed by the light. Then, if we start with light of intensity I_0 and allow it to traverse a medium for which the coefficient of transmission is constant and of value $\epsilon^{-\mu m}$, the resulting curve will be a logarithmic curve, and, if through the same medium light travels of initial intensity $I_0 \epsilon^{\mu n}$, the curve will again be a logarithmic curve lying above the other; then the curve of intensity, after traversing a medium of variable density such as we are now considering, will be a sinuous curve always situated between the above-mentioned curves and touching them alternately, the points of contact with the upper curve corresponding to those values of the abscissæ which make $\cos x$ vanish and the points of contact with the lower curve corresponding to those values of the abscissæ which make $\cos x$ equal to unity.

IV. By means of the general expression for the intensity of transmitted light we may also solve inverse questions such as What must be the law of density in order that the intensity may be some assigned function of the distance traversed? Suppose, for instance, we require the

law of density in order that the intensity may vary as some inverse power of the distance: in this case

$$\mathrm{I}=\frac{m}{t^n},$$

so that the general equation becomes

$$\frac{m}{t^n}=\mathrm{C}\epsilon^{-\mu\chi(t)}.$$

Differentiating, we obtain

$$-\frac{mn}{t^{n+1}}=-\mathrm{C}\epsilon^{-\mu\chi(t)}\phi(t)\mu,$$

since

$$\frac{d\chi(t)}{dt}=\phi(t).$$

But

$$\phi(t)=d,$$

Hence we get the following equation:—

$$\frac{n}{t}=\mu d,$$

so that d varies as the distance inversely. Since

$$\mathrm{I}=\frac{m}{t^n},$$

we get, by eliminating t,

$$\mathrm{I}=m\left(\frac{\mu}{n}\right)^n d^n;$$

this gives the relation between the density and the intensity at any point.

Note.—In several papers on colorimetry which I have read before this Society, I have frequently had occasion to refer to the hypothesis that as the length of an absorbing

medium increased according to the terms of an arithmetical ratio, the intensity of the light diminishes according to the terms of a geometrical ratio. I had not then been able to trace this hypothesis back farther than the writings of Sir John Herschel, but had some grounds for supposing that it might have been given earlier, and more especially by Bouguer. Lately, after much inquiry, there has come into my hands a small treatise, entitled 'Essai d'Optique sur la Gradation de la Lumière, par M. Bouguer, professeur royal en Hydrographie,' Paris, 1729. From this work it appears that the honour of having first announced the hypothesis belongs to Bouguer.

In many otherwise excellent treatises on Physics and Optics the subject of absorption of light is either neglected or scantily treated, and the claims of Bouguer seem to have nearly passed out of recognition; yet he may assuredly claim herein a position correlative with that assigned to Snell, or Huyghens, or Newton in those departments of optics of which they laid the foundations. The treatise contains no experimental verification of the hypothesis, nor any suggestions for carrying out such experiments. He was aware that the subject afforded a vast field for future inquiries, and with regard to his own work he modestly states, in the preface, "C'est vrai que mes recherches sont poussées si peu loin qu'elles laissent encore un vaste champ à tous ceux qui voudront perfectionner cette matière. Mais ne sçait-on pas que les Arts les plus simples ont eu leurs différens âges, et que ce seroit comme étoufer dans le berceau les découvertes qu'on peut faire dans la suite, que de mépriser toutes les premières tentatives, sous prétexte que ce ne sont encore que de foibles commencemens?"

In the last section he has also considered the intensity of light which has passed through a medium which is not

of the same density throughout. By geometrical reasoning he arrives at the conclusion that the curve of intensity (the *gradulucique,* as he terms it) has this property: its subtangent multiplied by the density is equal to a constant. Expressed in the language of the differential calculus, this gives rise to a differential equation similar to the one which I obtained by a different method, and gave last session in a paper read before the Society on the intensity of light which has traversed a medium wherein the density is some function of the distance traversed. Except in the consideration of the intensity of light which has passed through the atmosphere, Bouguer has made very little use of this highly general theorem; for, says he, in most cases we do not know what is the law of density. This may be so, but by assuming the density to be some function of the distance, we may deduce some interesting and valuable results.

XX. *On a New Variety of Halloysite from Maidenpek, Servia.* By H. E. Roscoe, LL.D., F.R.S., President.

Read January 22nd, 1884.

This mineral, one of the few peculiar to Servia, was given to me by Mr. James Taylor, lately a resident at Maidenpek. It is a very soft (h=2·5) whitish green non-crystalline mineral, having a conchoidal waxy fracture. It is translucent in thin films, but opaque in mass, and adherent to the tongue. Its specific gravity is 2·07. On exposure to air it loses a portion of its combined water,

and becomes of a dead white colour and more opaque. The greenish tint is due to the presence of small quantities of copper oxide (1·11 per cent.).

The following analyses show that this is a more highly hydrated variety than most of the specimens of Halloysite hitherto examined, and that it corresponds to the formula $Al_2O_3 2SiO_2 + 5H_2O$.

Analysis of Halloysite from Maidenpek :—

	I.	II.	Mean.
Al_2O_3	32·81	32·58	32·69
SiO_2	37·59	37·70	37·64
H_2O	28·59	28·27	28·43
CuO	1·11	—	1·11
	100·10	—	99·87

The calculated composition for the above formula is :—

Al_2O_3	32·86
SiO_2	38·37
H_2O	28·77
	100·00

A specimen of a similar mineral from the same locality was found by Tietze to contain :—

$Al_2O_3(Fe_2O_3)$	25·20
SiO_2	44·96
H_2O	29·50
	99·66

Corresponding nearly with the formula $Al_2O_3 3SiO_2 + 6H_2O$. Showing a distinct difference in the relation of alumina to silica from that existing in the specimen in question.

XXI. *On the Introduction of Coffee into Arabia.*

By C. Schorlemmer, F.R.S.

Read February 5th, 1884.

In two papers, which I read on April 3rd and October 16th, before this Society, I mentioned that the custom of drinking coffee originated with the Abyssinians, who cultivated the plant from time immemorial. In Arabia it was not introduced until the early part of the fifteenth century; before this time the beverage made from the leaves of the kát was generally used, and is still in use.

A few weeks ago I received a letter from Professor W. T. Thiselton Dyer, F.R.S., in which he says: "Possibly the enclosed extracts from an old book of the last century may interest you.

"The point is that the introduction of the use of coffee from Persia, in the 15th century, seems to have led to the neglect of kát.

"Your interesting observation as to the absence of caffeine in the latter, would perhaps show that the change from one to the other had a physiological significance."

This appears very plausible. I hope to be able to obtain a larger supply of kát, in order to find out its active principle.

The extracts which Professor Dyer sent me are as follows:—

'A Historical Treatise of the Original of Coffee.' London, 1732 (pp. 308–310).

"Jem al Adin Abu Abdallah, Mohammed Bensaid, surnamed Al Dhabhani (because he was a native of Dhabhan, a small town of Arabia Fœlix), being Mufti* of Aden, a famous town, and part of the same country, about the middle of the 9th age of the Hegirah, and of the 15th of our Lord, had occasion to make a voyage to Persia. During his stay there, he found some of his countrymen who took coffee†, which, at first, he took no great notice of; but at his return to Aden, his health being impaired, and calling to mind the coffee which he had seen taken in Persia, he took some in hopes it might do him good. Not only the Mufti's health was restored by the use of it, but he soon became sensible of the other properties of coffee; particularly that it dissipates heaviness in the head, exhilarates the spirits, and hinders sleep without indisposing one."

The Arabian author adds, that they found coffee so good that they entirely left off the use of another liquor which was in vogue at Aden, made of the leaves of a plant called *Cat*, which cannot be supposed to be the *The*, because this writer says nothing which might favour that opinion.

Since this was written Mr. W. Elborne, of the Owens College, called my attention to a paper by Mr. James Vaughan, Civil and Port Surgeon at Aden, who states that some estimate may be formed of the strong predilection which the Arabs have still for kát, from the quantity used in Aden alone, which averages about 280 camel-loads annually. He adds that he is not aware that kát is used in Aden in any other way than for mastication; from what he has heard, however, he believes a decoction

* An order of Priests amongst the Mahometans, which may be called their Bishops.

† Coffee first in use at Aden, the capital city of Arabia Fœlix.

resembling tea is made from the leaf by the Arabs in the interior*.

Mr. Vaughan gives also some abstracts from De Lacy's 'Chrestomathie arabe,' in which it is stated, on the authority of some Arabian authors, that coffee was not introduced into Arabia by Mohammed Dhabhani, as it was generally stated, but by the learned and godly Ali Shadeli ibn Omar. In the days of Mohammed Dhabhani, kát, which previous to that time was used, had disappeared from Aden. "Then it was that the Sheik advised those who had become his disciples to try the drink made from the *boonn* (coffee-berry), which was found to produce the same effect as the kát, including sleeplessness, and that it was attended with less expense and trouble. The use of coffee has been kept up from that time to the present."

As the custom of drinking coffee originated in Abyssinia, it appears more probable that it was introduced into Arabia from this country, and not from Persia.

My friend Professor Theodores has informed me that the beverage made from the boonn is called *kahwa*. This word is derived from *ikha*, dislike or distaste, *i. e.* for eating and sleeping.

* Pharm. Journ. Trans. xii. p. 268 (1852–53).

XXII. *On the Equations and on some Properties of Projected Solids.* By JAMES BOTTOMLEY, B.A., D.Sc., F.C.S.

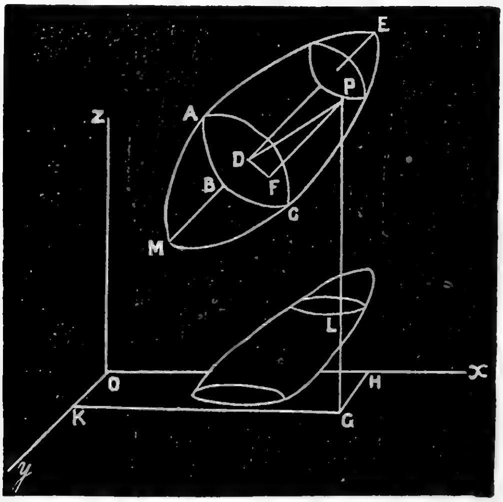
Read March 18th, 1884.

ON a former occasion I brought before the Physical and Mathematical Section of this Society a proposition in projection, in which it was shown how, by the composition of two projections, namely, of that of a line on a line, and of that of a plane area on a plane area perpendicular to the aforesaid line, we could derive from a solid three solids with axes perpendicular to three planes at right angles and of variable volume, the variation being subject to the condition that the sum of the three volumes is constant and equal to that of the primitive solid.

I now propose to solve the following problem: Given the equation to the primitive solid to deduce that of a derived solid.

Let the equation to the primitive solid referred to three rectangular axes be

$$f(x, y, z) = 0.$$



Let ABC be the primitive plane, which is fixed in the solid, and DE an axis perpendicular to this plane, and which may be called the primitive axis. Let P be a point situated on the intersection of the solid by a plane parallel to the primitive plane. Draw PF perpendicular to that plane.

In deducing the equations to the derived solid we might have taken a system of axes in the primitive solid as follows: take DE as one axis, and two straight lines perpendicular to this, lying in the plane ABC and passing through the point D, as the other two axes. Then, if the equation to the solid be given, referred to these movable axes, we may deduce by geometry the coordinates of any point on the derived solid referred to the axes fixed in space Ox, Oy, Oz. We may, however, refer both the derived and the primitive solid to the same system of fixed axes.

Draw PG perpendicular to the plane xy; on PG take a length LG, so that

$$\text{LG}=\text{PF}\cos\gamma,$$

γ being the inclination of the primitive axis to the axis of z. Then L will be a point on the derived solid. Also we have

$$\text{PF}=\text{PD}\cos\text{DPF}.$$

DPF is the angle between PF and PD. PF is parallel to the primitive axis, and its direction cosines will therefore be $\cos\alpha$, $\cos\beta$, $\cos\gamma$. Let a, b, c be the coordinates of the point D; then the direction cosines of the line DP will be

$$\frac{x-a}{\text{PD}},\quad \frac{y-b}{\text{PD}},\quad \frac{z-c}{\text{PD}}.$$

Therefore we have

$$\cos\text{DPF}=\frac{(x-a)\cos\alpha+(y-b)\cos\beta+(z-c)\cos\gamma}{\text{PD}};$$

also we have

$$x = \mathrm{KG},$$
$$y = \mathrm{HG},$$
$$z = \mathrm{PG}.$$

Let ξ, η, ζ denote the coordinates of the corresponding point on the derived solid, thus

$$\xi = \mathrm{KG},$$
$$\eta = \mathrm{HG},$$
$$\zeta = \mathrm{LG} = \mathrm{PD} \cos \mathrm{DPF} \cos \gamma.$$

Hence we obtain

$$\left.\begin{aligned} &\xi = x, \\ &\eta = y, \\ &\zeta = \cos\gamma\{(x-a)\cos\alpha + (y-b)\cos\beta + (z-c)\cos\gamma\}. \end{aligned}\right\} \quad (1)$$

Hence the equation to the derived solid will be

$$f\left(\xi,\ \eta,\ \frac{\zeta}{\cos^2\gamma} - \frac{(\xi-a)\cos\alpha + (\eta-b)\cos\beta}{\cos\gamma} + c\right) = 0.$$

If z be given as an explicit function of x and y, say

$$z = \phi(x, y),$$

then the derived surface will be

$$\zeta = \cos\gamma\{(\xi-a)\cos\alpha + (\eta-b)\cos\beta - c\cos\gamma\} + \cos^2\gamma\phi(\xi, \eta).$$

Hence the equation of the derived solid will be of the same degree as the primitive solid.

In the figure we have taken as primitive plane, a plane intersecting the solid, and its projection as falling on the plane x, y; in this case that portion of the solid lying below the primitive plane will, when projected, lie below the plane x, y. We might have taken as the primitive plane the tangent plane at the lower extremity of the primitive axis; in this case the whole of the solid, when projected, would be above the plane x, y.

As an example of the use of the above given relationships between the coordinates of the primitive and the derived solid, suppose we find the relation between their volumes. Then using V_z to denote the volume of the derived solid between limits 0 and ζ, we have

$$V_z=\iint\zeta\, d\eta\, d\xi\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

by substitution this becomes

$$V_z=\cos\gamma\iint\{(x-a)\cos\alpha+(y-b)\cos\beta+(z-c)\cos\gamma\}\,dy\,dx,$$

a result which may also be written in the form

$$V_z=\cos^2\gamma\iint\left(z-c+\frac{(x-a)\cos\alpha+(y-b)\cos\beta}{\cos\gamma}\right)dy\,dx,$$

or finally in the form

$$V_z=\cos^2\gamma\iiint_{c-\frac{(x-a)\cos\alpha+(y-b)\cos\beta}{\cos\gamma}}^{z} dx\,dy\,dz.$$

If, in equation (1), we make $\zeta=0$, we obtain

$$(x-a)\cos\alpha+(y-b)\cos\beta+(z-c)\cos\gamma=0.$$

This, being an equation of the first degree in x, y, z, will represent a plane; it is, in fact, the equation to the primitive plane. Hence the lower limit in the above integral reminds us that the integration is to extend from the primitive plane to points upwards; but if V be the volume of the primitive solid between these limits we shall have

$$V=\iiint_{c-\frac{(x-a)\cos\alpha+(y-b)\cos\beta}{\cos\gamma}}^{z} dx\,dy\,dz.$$

Hence

$$V_z=\cos^2\gamma V.$$

In a similar manner we might show that

$$V_x = \cos^2\alpha V,$$
$$V_y = \cos^2\beta V,$$

and so establish the relation given in a previous paper,

$$V_z + V_y + V_x = V.$$

In equation (2) suppose that the limits of the integration are ζ_1 and ζ_2, so that the volume of the solid is $\iint(\zeta_2-\zeta_1)\,d\xi d\eta$. Let z_1 and z_2 be the corresponding values of the points on the primitive surface. Now, in integrating with respect to z, we regard x and y as constant; hence

$$\zeta_2 = \cos\gamma\{(x-a)\cos\alpha + (y-b)\cos\beta + (z_2-c)\cos\gamma\},$$
$$\zeta_1 = \cos\gamma\{(x-a)\cos\alpha + (y-b)\cos\beta + (z_1-c)\cos\gamma\};$$

by subtraction we obtain

$$\zeta_2 - \zeta_1 = (z_2 - z_1)\cos^2\gamma;$$

hence, by substitution, we obtain

$$\iint(\zeta_2-\zeta_1)\,d\xi\,d\eta = \cos^2\gamma\iint(z_2-z_1)\,dx\,dy.$$

As a particular case, consider the projected solid derived from a sphere

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

Take as primitive plane a plane passing through the centre of the sphere, and as primitive axis a line perpendicular to this passing through the centre of the sphere; then the equation of the projected solid will be

$$n^4\{(x-a)^2 + (y-b)^2\} + \{z - n(l(x-a) + m(y-b))\}^2 = n^4 r^2, \quad (3)$$

l being written for $\cos\alpha$, m for $\cos\beta$, n for $\cos\gamma$.

To simplify the equation to this surface remove the

origin to the point a, b, o, and let the new axis of x make an angle θ with the old axis, so that

$$\tan\theta = \frac{m}{l};$$

then the equation becomes

$$z^2 - 2znx\sqrt{1-n^2} + n^2x^2 + n^4y^2 = n^4r^2.$$

Now transform to new axis in the plane of x, z, the new axis of x making, with the old, an angle ϕ, so that

$$\tan 2\phi = \frac{2n}{\sqrt{1-n^2}};$$

then we get the equation

$$\frac{\dfrac{x^2}{2n^4r^2}}{1+n^2-\sqrt{1+2n^2-3n^4}} + \frac{\dfrac{z^2}{2n^4r^2}}{1+n^2+\sqrt{1+2n^2-3n^4}} + \frac{y^2}{r^2} = 1. \quad (4)$$

Thus the surface is an ellipsoid; its volume may readily be shown to be $\frac{4}{3}\pi n^2r^3$, that is $n^2\times$ vol. of sphere. If $n=\pm 1$, that is if the primitive axis be parallel to the axis of z, the equation to the ellipsoid becomes

$$x^2+y^2+z^2=r^2,$$

which is the equation to the primitive solid referred to coordinates passing through its centre.

If, in equation (4), we make $n=0$, the principal axis of the ellipsoid parallel to the axis of x will assume the indeterminate form $\frac{0}{0}$. On examination it will be found to be r^2, but when $n=0$ the projected solid vanishes; hence, just as it vanishes the ellipsoid becomes a spheroid, of which one of the axes is indefinitely small.

In a former communication I stated that a sphere when

projected would give a spheroid of which the semiaxes are R, R cos γ, R cos γ, whereas in (3) an ellipsoid has been obtained. It will be observed, however, that there is something arbitrary in the mode of projection. The sole condition to which the projected solid is subjected is

$$\int A dz = \cos^2\gamma V;$$

but this condition is not sufficient to determine the equation to the projected solid. If L be the length of the primitive axis, then if we draw two parallel planes distant L cos γ, these planes being parallel to the plane of x, y, any solids terminated by these planes and having equal sections made by any plane parallel to them, will fulfil the above condition. Hence we have some choice as to the manner of laying the successive slices of the projected solid on one another; we have just found an ellipsoid as a particular case, but if the elliptical sections parallel to the plane of x, y had been piled on one another, so that all the centres lay in the same vertical line, we should obtain a spheroid of which the semiaxes are R, R cos γ, R cos γ.

The locus of the centres in (3) may be got as follows. Let a section be made by a plane parallel to plane of x, y and distant ζ from that plane; write also $X+X_1$ for x, and $Y+Y_1$ for y. Let X_1 and Y_1 be so determined that the coefficients of X and Y vanish. This leads to the condition

$$(X_1-a)(n^4+n^2l^2) + n^2lm(Y_1-b) - \zeta nl = 0,$$
$$(Y_1-b)(n^4+n^2m^2) + n^2lm(X_1-a) - \zeta nm = 0.$$

Hence, X_1, Y_1, Z_1 being the coordinates of the centre of the section, we have

$$n(Y_1-b) = \zeta m,$$
$$n(X_1-a) = \zeta l,$$
$$Z_1 = \zeta.$$

From these equations we obtain

$$\frac{X_1-a}{l}=\frac{Y_1-b}{m}=\frac{Z_1}{n}.$$

The locus of the centres is therefore a straight line parallel to the primitive axis; we might have drawn any arbitrary curve, either plane or of double curvature, terminated by the bounding planes parallel to plane of x, y, and piled up the successive slices, so that their centres lay on the curve; the solid so generated would also fulfil the condition of a projected solid.

As another example, consider the projection of the elliptic paraboloid

$$z=Ax^2+By^2.$$

The equation to the derived solid is

$$z=n\{(x-a)l+m(y-b)-cn\}+n^2(Ax^2+By^2).$$

The volume of the primitive solid included between the curved surface and the plane

$$z=h$$

will be

$$\iiint_{Ax^2+By^2}^{h} dx\,dy\,dz.$$

The plane, when projected, will give an equation of the form

$$z=n\{l(x-a)+m(y-b)-nc\}+n^2h.$$

The volume intercepted between this plane and the projected solid will be

$$\iiint_{n\{l(x-a)+m(y-b)-nc\}+n^2(Ax^2+By^2)}^{n\{l(x-a)+m(y-b)-nc\}+n^2h} dx\,dy\,dz,$$

which is equivalent to

$$\iiint_{n^2(Ax^2+By^2)}^{n^2h} dx\,dy\,dz \quad \text{or to} \quad n^2\iiint_{Ax^2+By^2}^{h} dx\,dy\,dz,$$

that is $n^2\times$vol. of primitive solid between the corresponding limits.

The primitive solid and its z derivative being written in the form

$$z=\phi(x, y),$$
$$z=n\{l(x-a)+m(y-b)-cn\}+n^2\phi(x, y).$$

If we multiply the first equation by n^2 and subtract from the second, we get the equation

$$z(n^2-1)+nlx+nmy-n(la+mb+nc)=0.$$

Hence, if the primitive and its derivative have any points in common they satisfy the equation to a plane. This plane is at right angles to the primitive plane, for θ being the angle between the planes, we shall have

$$\begin{aligned}\cos\theta &= \frac{nl^2}{\sqrt{1-n^2}}+\frac{nm^2}{\sqrt{1-n^2}}-n\sqrt{1-n^2},\\ &= \frac{n}{\sqrt{1-n^2}}(l^2+m^2+n^2-1),\\ &= 0.\end{aligned}$$

The relation $V_z=\cos^2\gamma V$ is only a particular case of a more general theorem; for let $\phi(\xi, \eta)$ be any arbitrary function of ξ and η, and let I_z denote the integral

$$\iiint_0^{\zeta}\phi(\xi, \eta)\,d\xi\,d\eta\,d\zeta;$$

then, by substitution, we may show, I denoting the integral

$$\iiint_{c-\frac{(x-a)\cos\alpha+(y-b)\cos\beta}{\cos\gamma}}^{z} \phi(x, y)\, dx\, dy\, dz,$$

that

$$I_z = \cos^2\gamma I.$$

As another example of the use of the substituted coordinates, suppose that between any two points of the primitive solid any arbitrary curve be drawn lying on the surface. Let s be the length of this curve, and ds an element extending from the point x, y, z to the point $x+dx, y+dy, z+dz$. Let s_z, s_y, s_x be the lengths of the curves on the three derived solids passing through the points corresponding to those through which the curve on the primitive solid passes; then, ds_z, ds_y, ds_x being elements of these curves, we shall have

$$ds_z^2 = d\xi^2 + d\eta^2 + d\zeta^2;$$

by substitution this becomes

$$ds_z^2 = dx^2 + dy^2 + \cos^2\gamma\,(dx\cos\alpha + dy\cos\beta + dz\cos\gamma)^2,$$

and in like manner may be obtained

$$ds_y^2 = dx^2 + dz^2 + \cos^2\beta(dx\cos\alpha + dy\cos\beta + dz\cos\gamma)^2,$$
$$ds_x^2 = dy^2 + dz^2 + \cos^2\alpha\,(dx\cos\alpha + dy\cos\beta + dz\cos\gamma)^2.$$

By addition we obtain

$$ds_z^2 + ds_y^2 + ds_x^2 = 2\,(dx^2 + dy^2 + dz^2) + (dx\cos\alpha + dy\cos\beta + dz\cos\gamma)^2. \quad (5)$$

Let ϕ be the angle between the primitive axis and the

tangent at any point to the fixed curve on the primitive solid, then we have

$$\cos\phi = \frac{dx}{ds}\cos\alpha + \frac{dy}{ds}\cos\beta + \frac{dz}{ds}\cos\gamma,$$

also

$$ds^2 = dx^2 + dy^2 + dz^2 ;$$

hence the right side of (5) may be written in the form $ds^2(2+\cos^2\phi)$. Since the curve is fixed on the solid, if we suppose the solid to move in any way, we shall have $\int\sqrt{2+\cos^2\phi}\,.\,ds$ a constant quantity; if this be denoted by C, then we shall have throughout the motion

$$\int\sqrt{ds_z^2 + ds_y^2 + ds_x^2} = \mathrm{C}.$$

The lengths of the curves on the derived solids will vary as the primitive solid changes its position, but will vary in such a manner as to satisfy the above equation.

By reference to the expression for ds_z, it will be seen that it does not vanish when $\cos\gamma$ vanishes; under this last condition the dimensions of V_z perpendicular to the plane of x, y become indefinitely small; in other words, the solid degenerates into a plane area, simultaneously the projected curve degenerates into a curve of single curvature lying on this plane area.

The equation to the primitive solid being given in the form

$$z = \phi(x, y),$$

then the superficial area of this solid will be given by the equation

$$\mathrm{S} = \iint\sqrt{1 + \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2}\,dx\,dy ;$$

the superficial area of the z derivative will be given by the equation

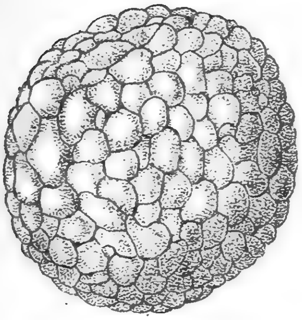
$$S_z = \iint \sqrt{1 + n^2\left(l + n\frac{d\phi}{dx}\right)^2 + n^2\left(m + n\frac{d\phi}{dy}\right)^2}\, dx\, dy.$$

When $n=1$, and therefore $m=0$, $l=0$, we have

$$S = S_z.$$

When $n=0$ it will be observed that S_z does not vanish; the equation then assumes the form

$$S_z = \iint dx\, dy.$$



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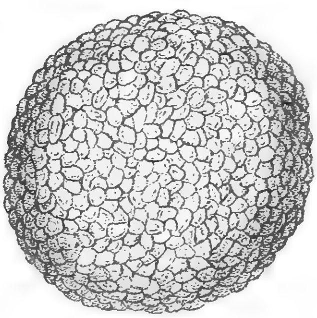

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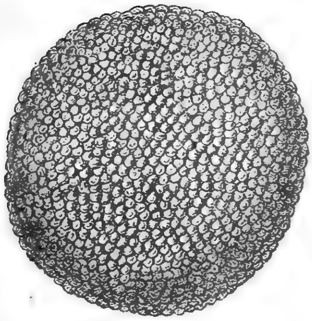

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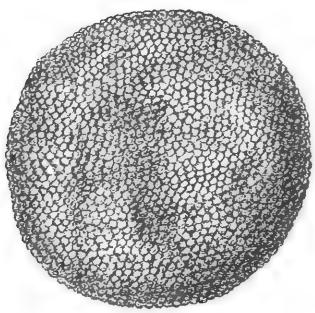

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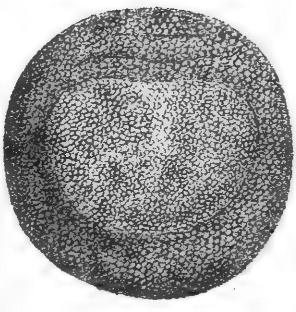

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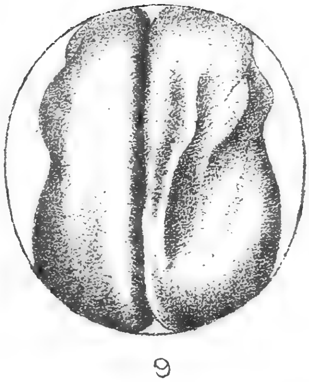

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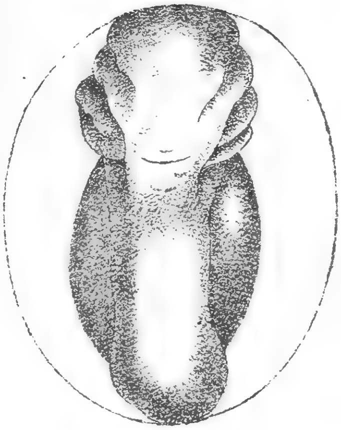
Dr. Alcock: "On the Development of the Common Frog."



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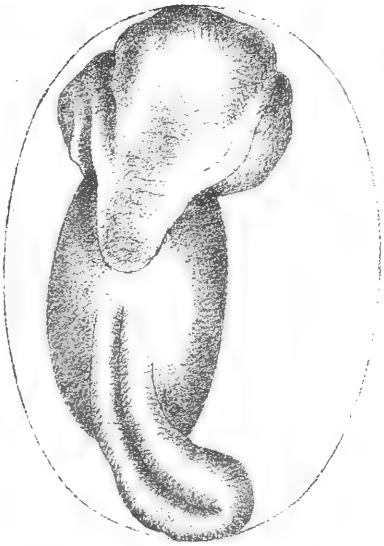

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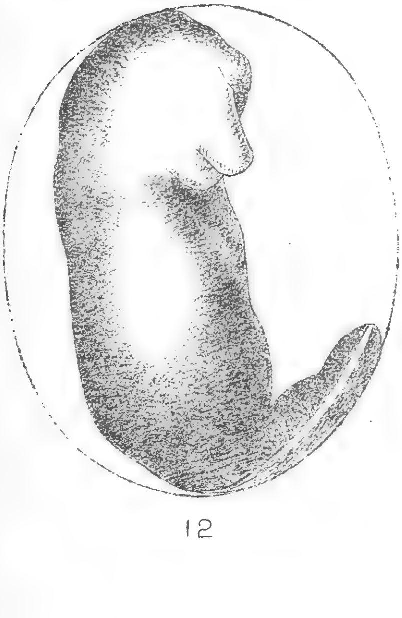

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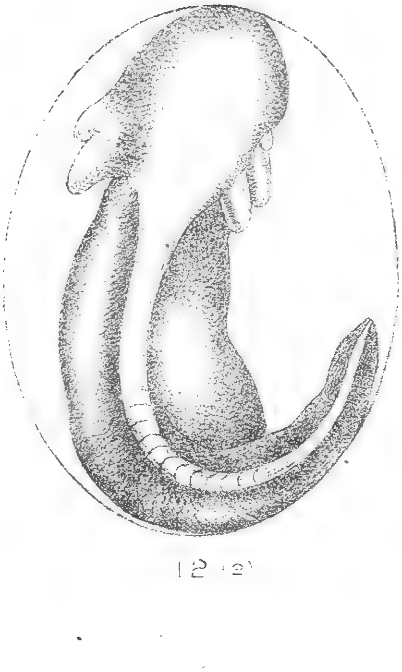

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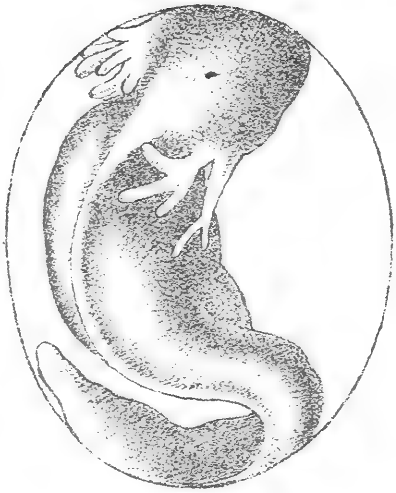

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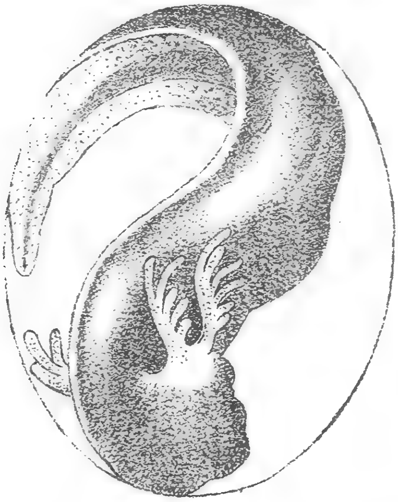

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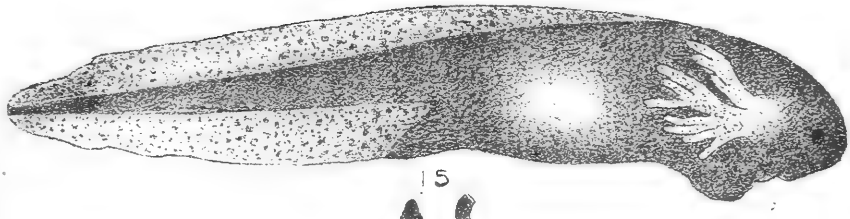

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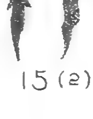

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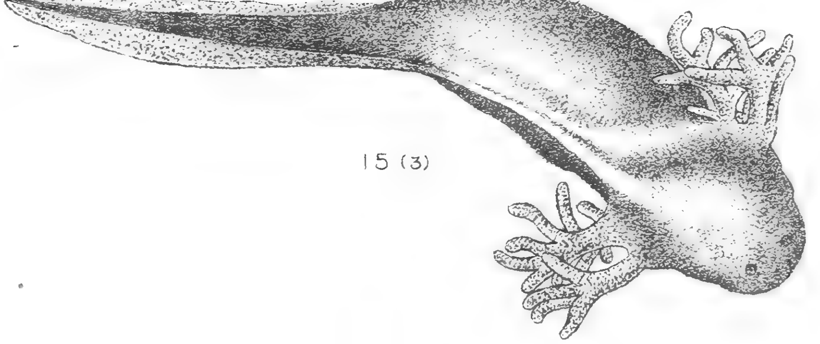

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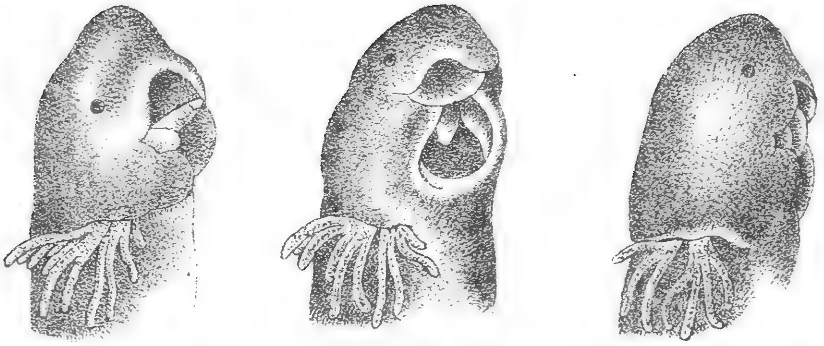

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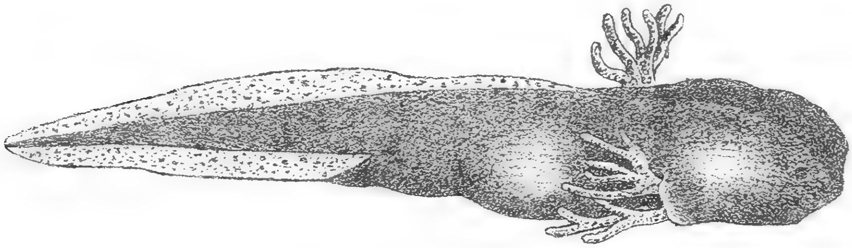

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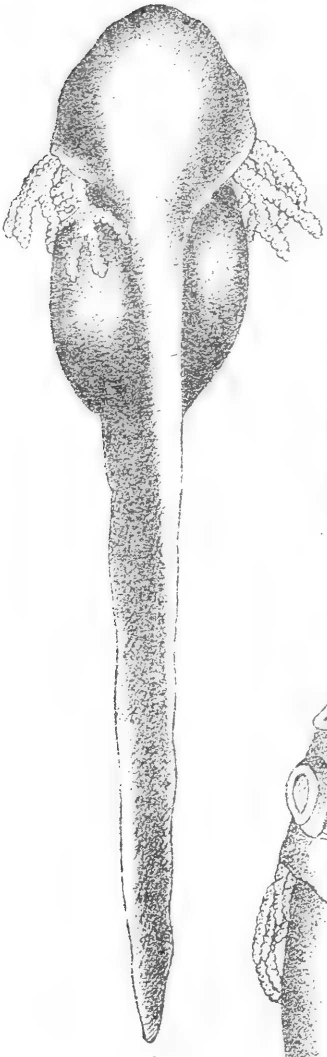

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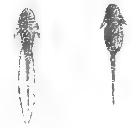

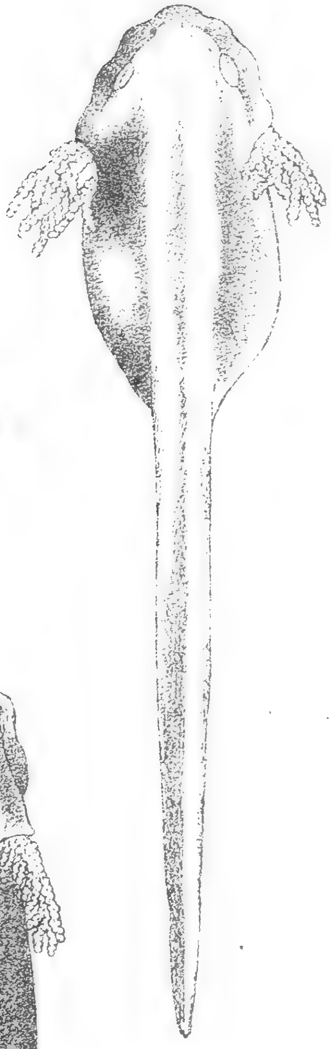

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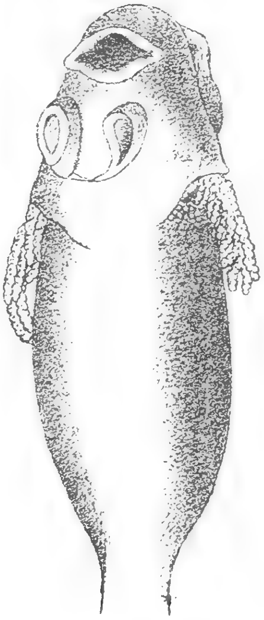

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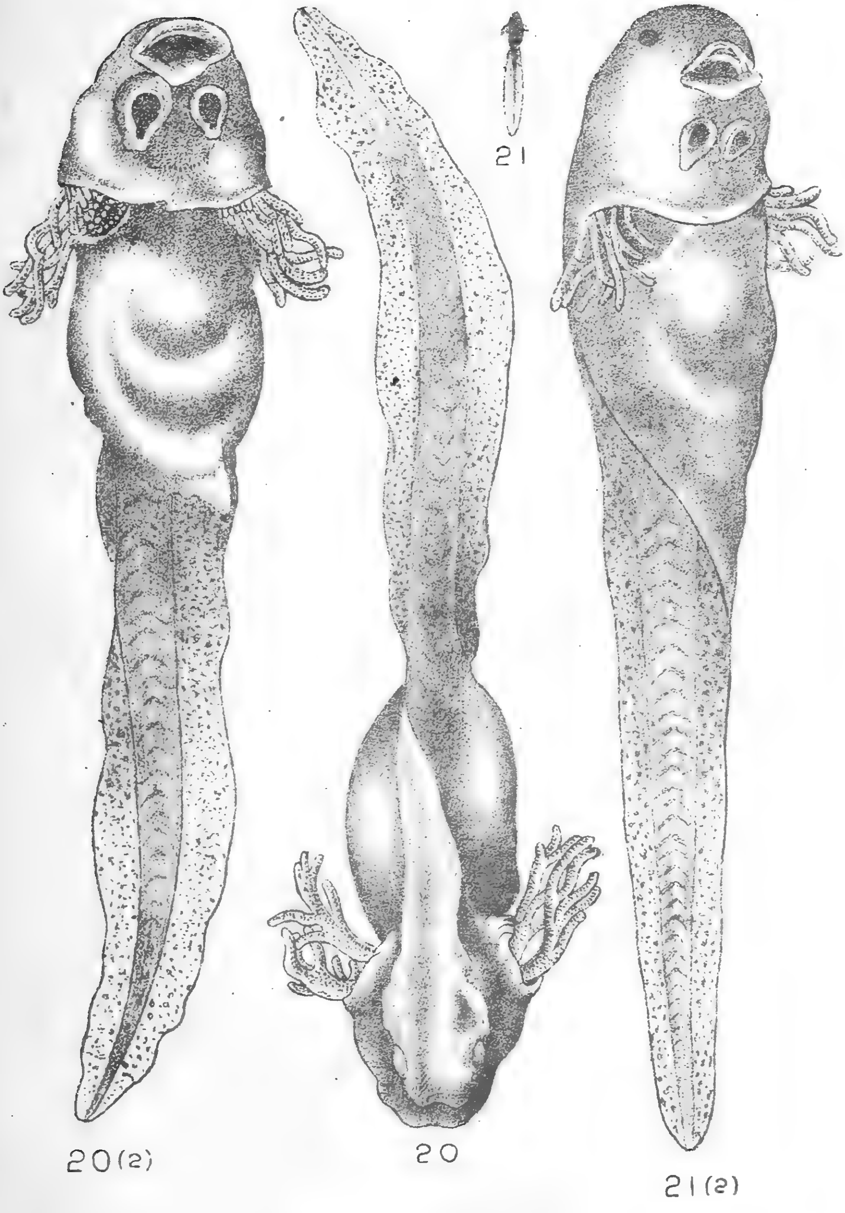
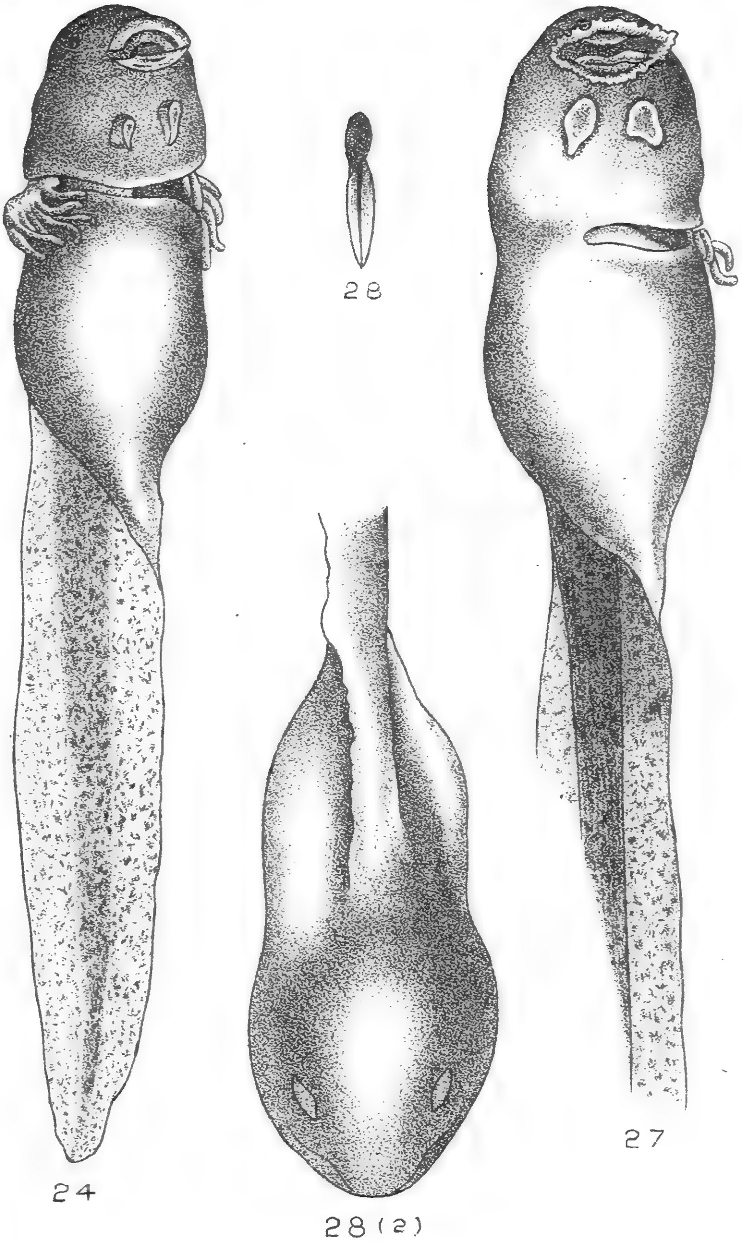
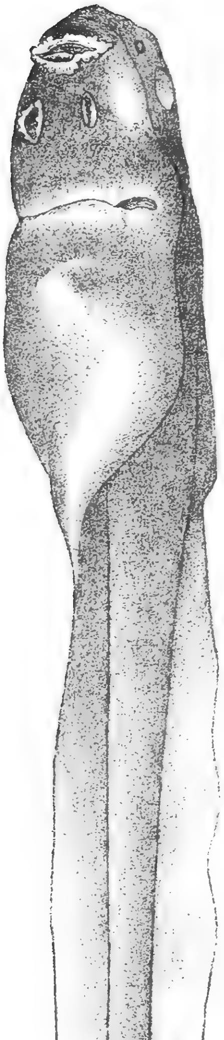


Dr. Alcock: "On the Development of the Common Frog."



Dr. Alcock: "On the Development of the Common Frog."



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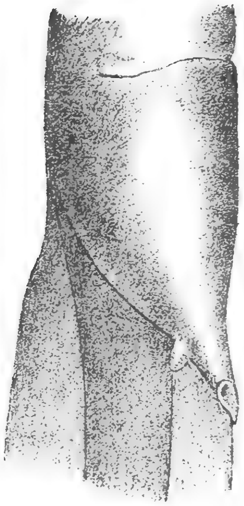

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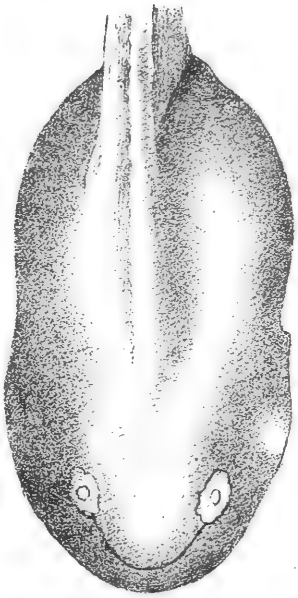

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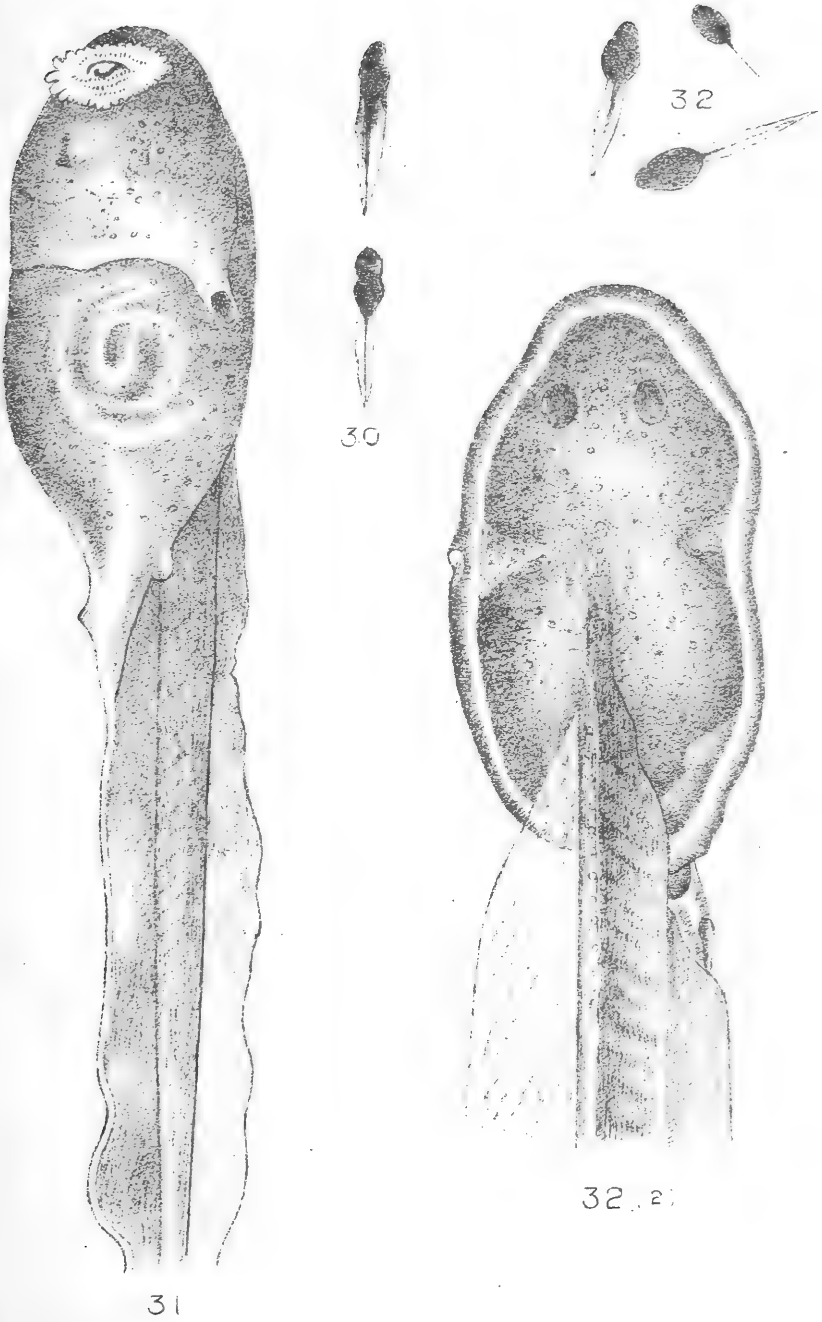

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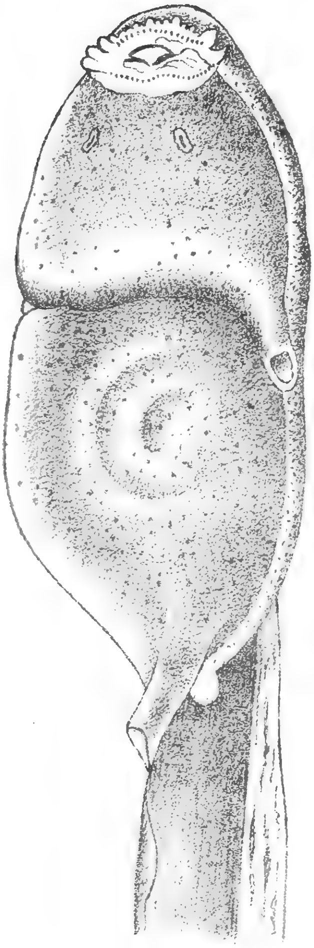

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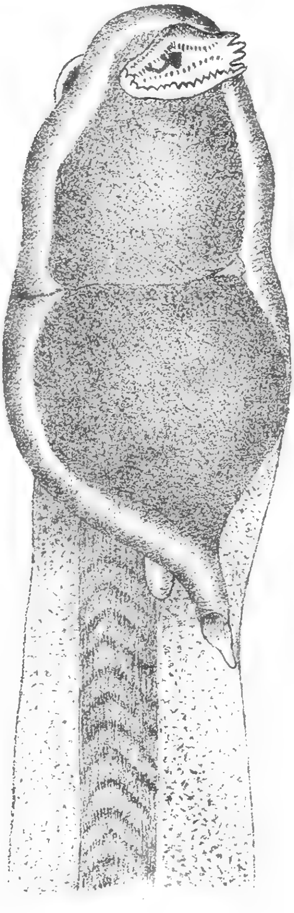

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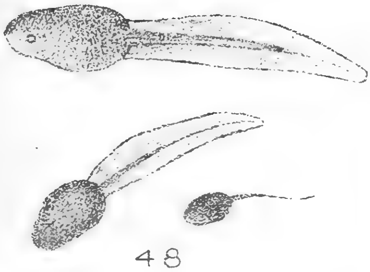

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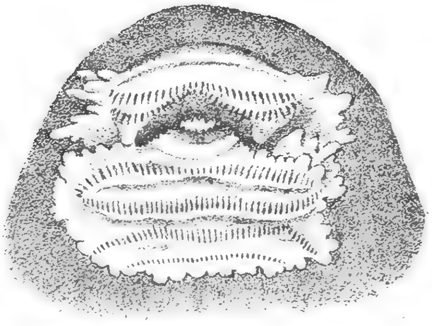

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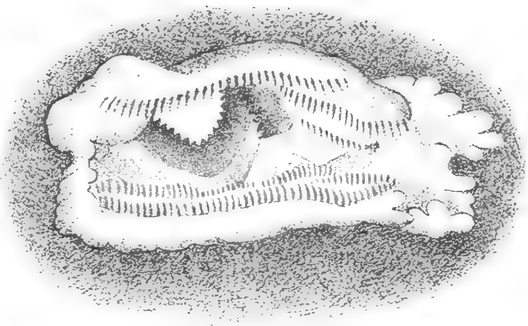

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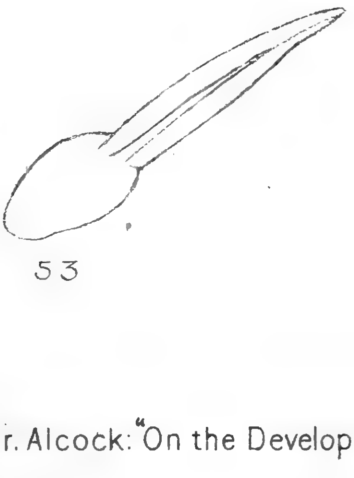

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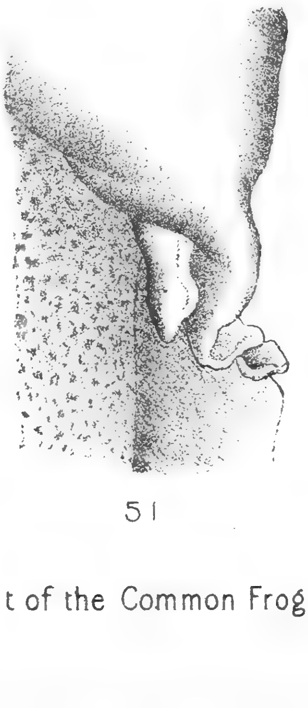

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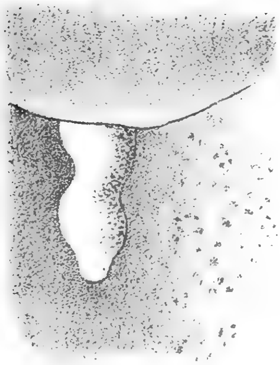
Dr. Alcock: "On the Development of the Common Frog".



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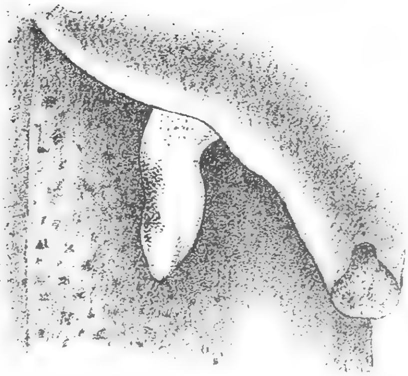

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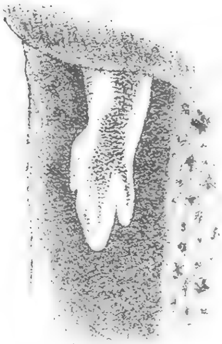

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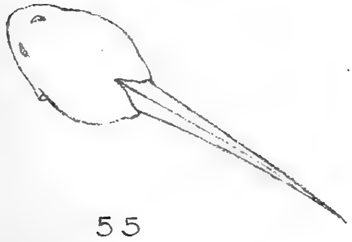

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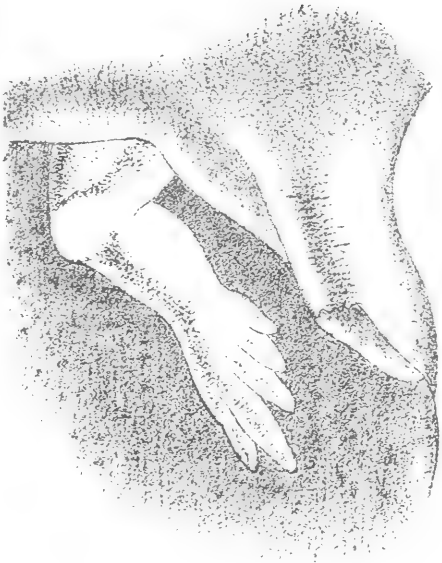

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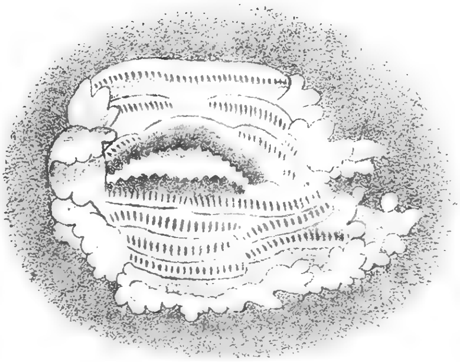

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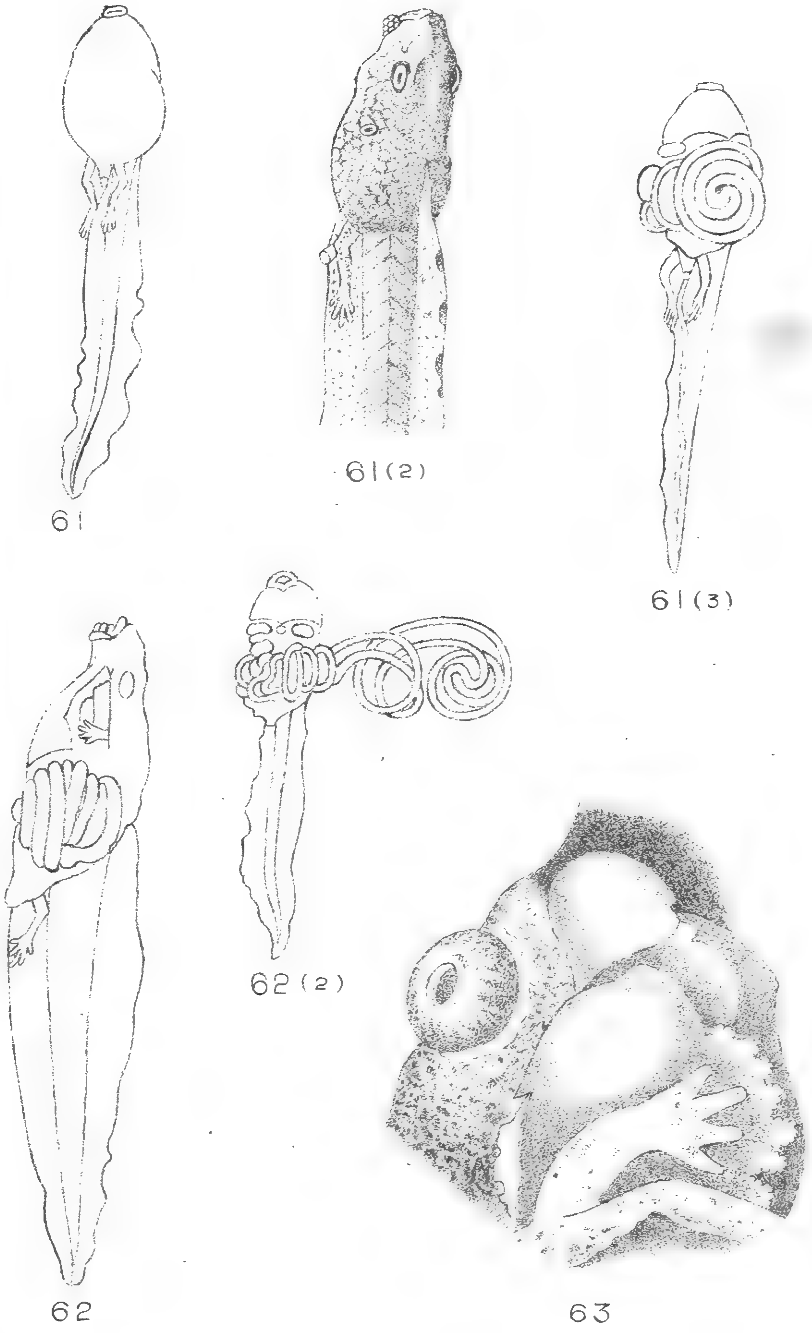
Dr. Alcock: "On the Development of the 'Common Frog."



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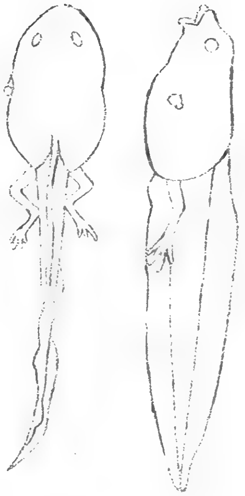

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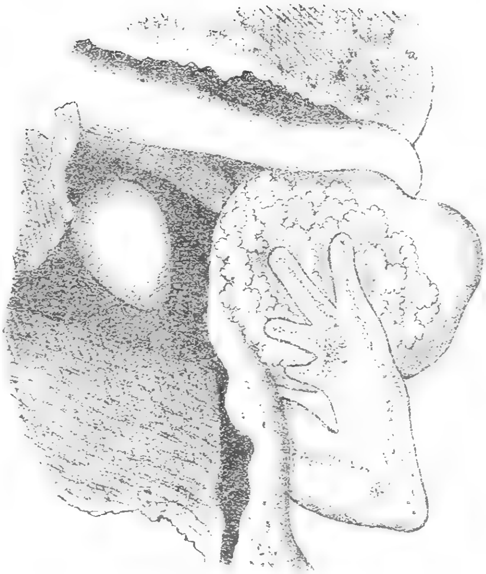

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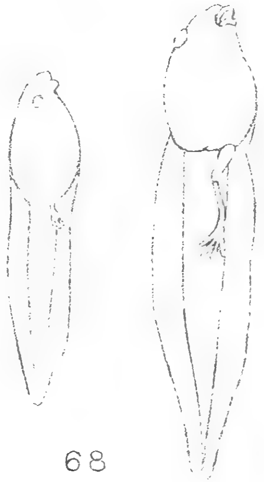

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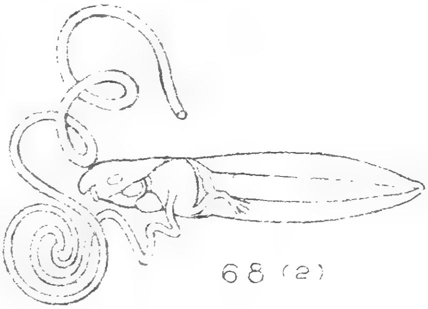

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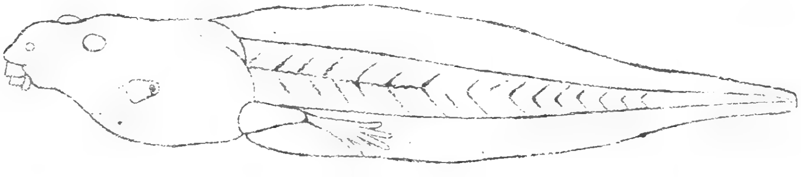

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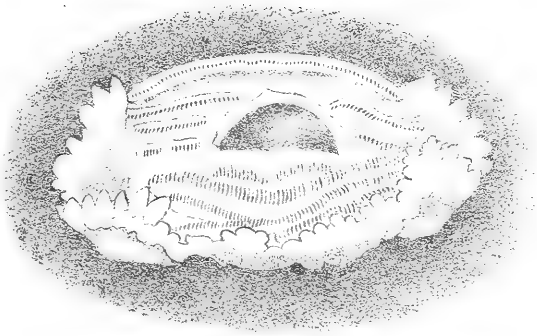

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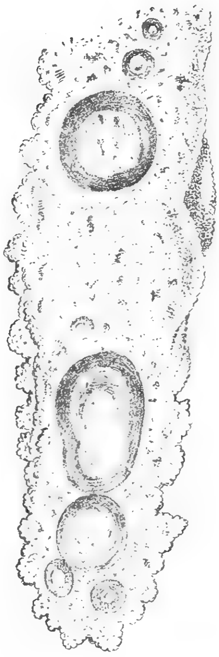

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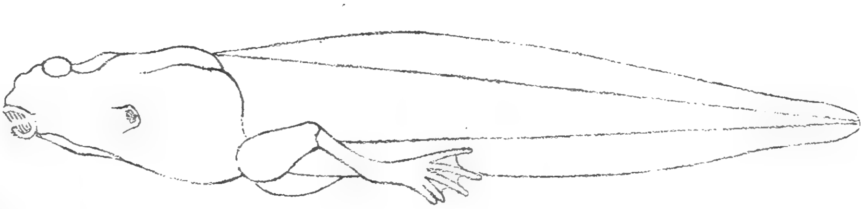

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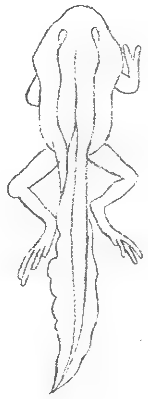

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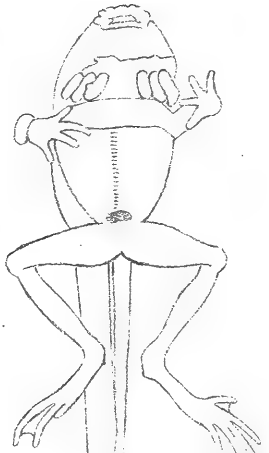

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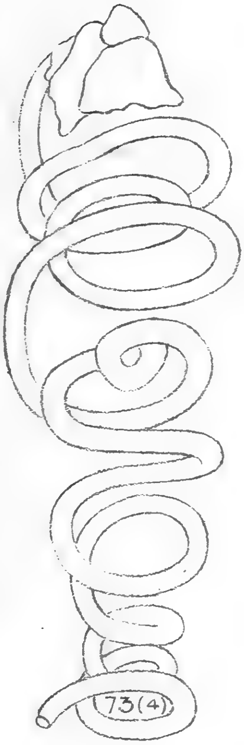

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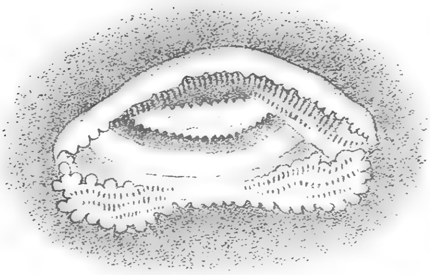

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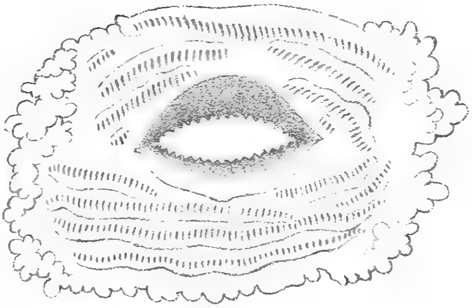

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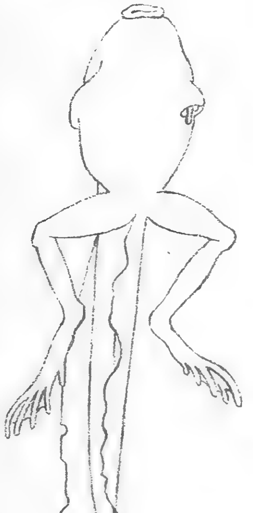

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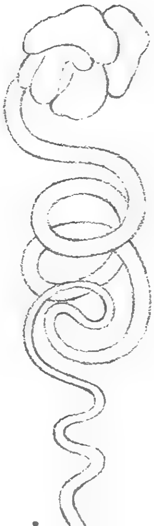

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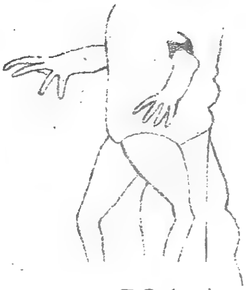

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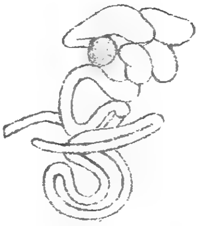

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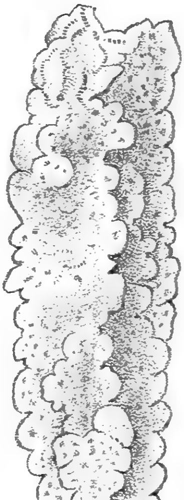

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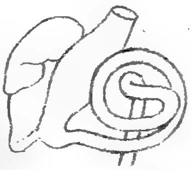

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1853, Apr. 19.	Hartnup, John, F.R.A.S. *Observatory, Liverpool.*
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1866, Jan. 23. Hofmann, A. W., LL.D., Ph.D., F.R.S., F.C.S., Ord. Leg. Hon. S^{rum} Lazar. et Maurit. Ital. Eq., &c. 10 *Dorotheenstrasse, Berlin.*

1869, Jan. 12. Huggins, William, F.R.S., F.R.A.S. *Upper Tulse Hill, Brixton, London, S.W.*

1872, Apr. 30. Huxley, Thomas Henry, LL.D.(Edin.), Ph.D., Pres.R.S., Professor of Natural History in the Royal School of Mines, South Kensington Museum, F.G.S., F.Z.S., F.L.S., &c. *School of Mines, South Kensington Museum, S.W.*, and 4 *Marlborough-place, Abbey-road, N.W.*

1852, Oct. 16. Kirkman, Rev. Thomas Penyngton, M.A., F.R.S. *Croft Rectory, near Warrington.*

1844, Apr. 30. Owen, Sir Richard, K.C.B., M.D., LL.D., F.R.S., F.L.S., F.G.S., V.P.Z.S., F.R.C.S. Ireland, Hon. M.R.S.E., For. Assoc. Imper. Instit. France, &c. *Sheen Lodge, Richmond.*

1851, Apr. 29. Playfair, Rt. Hon. Lyon, C.B., Ph.D., F.R.S., F.G.S., M.P., F.C.S., &c. 68 *Onslow-gardens, London, S.W.*

1866, Jan. 23. Prestwich, Joseph, F.R.S., F.G.S. *Shoreham, near Sevenoaks.*

1866, Jan. 23. Ramsay, Sir Andrew Crombie, F.R.S., F.G.S., Director of the Geological Survey of Great Britain, Professor of Geology, Royal School of Mines, &c. *Geological Survey Office, Jermyn-street, London, S.W.*

1849, Jan. 23. Rawson, Robert, F.R.A.S. *Havant, Hants.*

1872, Apr. 30. Sachs, Julius, Ph.D. *Prague.*

1869, Dec. 14. Sorby, Henry Clifton, F.R.S., F.G.S., &c. *Broomfield, Sheffield.*

1851, Apr. 29. Stokes, George Gabriel, M.A., D.C.L., Sec. R.S., Lucasian Professor of Mathem. Univ. Cambridge, F.C.P.S., &c. *Lensfield Cottage, Cambridge.*

1861, Jan. 22. Sylvester, James Joseph, M.A., F.R.S., Professor of Mathematics. *Johns Hopkins University, Baltimore.*

1868, Apr. 28. Tait, Peter Guthrie, M.A., F.R.S.E., &c. Professor of Natural Philosophy, Edinburgh.

DATE OF ELECTION.

1851, Apr. 22. Thomson, Sir William, M.A., D.C.L., LL.D., F.R.SS. L. and E., For. Assoc. Imper. Instit. France, Prof. of Nat. Philos. Univ. Glasgow. 2 *College, Glasgow.*

1872, Apr. 30. Trécul, A., Member of the Institute of France. *Paris.*

1868, Apr. 28. Tyndall, John, LL.D., F.R.S., F.C.S., Professor of Natural Philosophy in the Royal Institution and Royal School of Mines. *Royal Institution, London, W.*

CORRESPONDING MEMBERS.

1860, Apr. 17. Ainsworth, Thomas. *Cleator Mills, near Egremont, Whitehaven.*

1861, Jan. 22. Buckland, George, Professor, University College, Toronto. *Toronto.*

1867, Feb. 5. Cialdi, Alessandro, Commander, &c. *Rome.*

1870, Mar. 8. Cockle, The Hon. Sir James, M.A., F.R.S., F.R.A.S., F.C.P.S. *Sandringham-gardens, Ealing.*

1866, Jan. 23. De Caligny, Anatole, Marquis, Corresp. Mem. Acadd. Sc. Turin and Caen, Socc. Agr. Lyons, Sci. Cherbourg, Liège, &c.

1861, Apr. 2. Durand-Fardel, Max, M.D., Chev. of the Legion of Honour, &c. 36 *Rue de Lille, Paris.*

1849, Apr. 17. Girardin, J., Off. Legion of Honour, Corr. Mem. Imper. Instit. France, &c. *Lille.*

1850, Apr. 30. Harley, Rev. Robert, F.R.S., F.R.A.S. *College Place, Huddersfield.*

1882, Nov. 14. Herford, Rev. Brooke. *Arlington-street Church, Boston, U.S.*

1862, Jan. 7. Lancia di Brolo, Federico, Duc., Inspector of Studies, &c. *Palermo.*

1859, Jan. 25. Le Jolis, Auguste-François, Ph.D., Archiviste perpétuel and late President of the Imper. Soc. Nat. Sc. Cherbourg, &c. *Cherbourg.*

DATE OF ELECTION.

1857, Jan. 27. Lowe, Edward Joseph, F.R.S., F.R.A.S., F.G.S., Mem. Brit. Met. Soc., &c. *Nottingham.*

1862, Jan. 7. Nasmyth, James, C.E., F.R.A.S., &c. *Penshurst, Tunbridge.*

1869, Jan. 12. Saint Venant, Barré de, Ingénieur en chef des Ponts et Chaussées, Corr. Soc. Sci. de Lille et de l'Acad. Romaine des *Nuovi Lincei*, &c.

1867, Feb. 5. Schönfeld, Edward, Ph.D., Director of the Mannheim Observatory.

1834, Jan. 24. Watson, Henry Hough. *Bolton, Lancashire.*

ORDINARY MEMBERS.

1881, Jan. 11. Adamson, Daniel, F.G.S. *The Towers, Didsbury.*

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1873, Jan. 7. Allman, Julius. 2 *Victoria-street.*

1870, Dec. 13. Angell, John. *Manchester Grammar School.*

1861, Jan. 22. Anson, Ven. Archd. George Henry Greville, M.A. *Birch Rectory, Rusholme.*

1882, Jan. 24. Arnold, William Thomas, M.A. *Plymouth Grove.*

1837, Aug. 11. Ashton, Thomas. 36 *Charlotte-street.*

1881, Nov. 1. Ashton, Thomas Gair, B.A. 36 *Charlotte-street.*

1874, Nov. 3. Axon, William E. A., M.R.S.L., F.S.I. *Fern Bank,* 1 *Bowker-street, Higher Broughton.*

1865, Nov. 15. Bailey, Charles, F.L.S. *Ashfield, College-road, Whalley Range.*

1883, Oct. 16. Baker, Harry, F.C.S. 262 *Plymouth-grove.*

1876, Nov. 28. Barratt, Walter Edward. *Kersal, Higher Broughton.*

1867, Nov. 12. Barrow, John. *Stafford-road, Ellesmere-park, Eccles.*

1858, Jan. 26. Baxendell, Joseph, F.R.A.S., Corr. Mem. Roy. Phys. Econ. Soc. Königsberg, and Acad. Sc. & Lit. Palermo. 14 *Liverpool-road, Birkdale, Southport.*

1847, Jan. 26. Bazley, Sir Thomas, Bart. *Riversleigh, Lytham.*

1877, Nov. 27. Becker, Wilfred, B.A. 4 *Commercial Buildings, Cross-street.*

DATE OF ELECTION.

1878, Nov. 26. Bedson, Peter Philips, D.Sc. *College of Science, Newcastle-upon-Tyne.*

1847, Jan. 26. Bell, William. 51 *King-street.*

1870, Nov. 15. Bennion, John A., B.A., F.R.A.S. *Barr Hill Mount, Bolton-road, Pendleton.*

1868, Dec. 15. Bickham, Spencer H., jun. *Endsleigh, Alderley Edge.*

1861, Jan. 22. Bottomley, James, D.Sc., B.A., F.C.S. 7 *Avenue, Lower Broughton.*

1875, Nov. 16. Boyd, John. 58 *Parsonage-road, Withington.*

1855, Apr. 17. Brockbank, William, F.G.S. *Prince's Chambers,* 26 *Pall Mall.*

1861, Apr. 2. Brogden, Henry, F.G.S. *Hale Lodge, Altrincham.*

1844, Jan. 23. Brooks, William Cunliffe, M.A., M.P. *Bank,* 92 *King-street.*

1860, Jan. 23. Brothers, Alfred, F.R.A.S. 14 *St.-Ann's-square.*

1846, Jan. 27. Browne, Henry, M.D., M.A., M.R.C.S. Engl. *Woodhey, Heaton Mersey.*

1872, Nov. 12. Burghardt, Charles Anthony, Ph.D. 110 *King-street.*

1874, Dec. 15. Carrick, Joseph. *Prince's Chambers,* 26 *Pall Mall.*

1842, Jan. 25. Charlewood, Henry. 1 *St. James's-square.*

1854, Apr. 18. Christie, Richard Copley, M.A., Chancellor of the Diocese. 2 *St. James's-square.*

1841, Apr. 30. Clay, Charles, M.D., Extr. L.R.C.P. Lond., L.R.C.S. Edin. 101 *Piccadilly.*

1853, Jan. 25. Cottam, Samuel, F.R.A.S. 6 *Essex-street.*

1859, Jan. 25. Coward, Edward. *Heaton Mersey, near Manchester.*

1861, Nov. 12. Coward, Thomas. *Bowdon.*

1848, Jan. 25. Crowther, Joseph Stretch. *Endsleigh, Alderley Edge.*

1876, Apr. 18. Cunliffe, Robert Ellis. 18 *Vine-grove, Pendleton.*

1861, Apr. 2. Cunningham, William Alexander. *Auchenheath Cottage, Lesmahagow, by Lanark, N.B.*

1854, Feb. 7. Dale, John, F.C.S. *Cornbrook Chemical Works, Chester-road.*

1871, Nov. 8. Dale, Richard Samuel, B.A. *Cornbrook Chemical Works, Chester-road.*

1853, Apr. 19. Darbishire, Robert Dukinfield, B.A., F.G.S. 26 *George-street.*

1878, Nov. 26. Davis, Joseph. *Engineer's Office, Lancashire and Yorkshire Railway, Hunt's Bank.*

1869, Nov. 2. Dawkins, William Boyd, M.A., F.R.S. *Museum, Owens College.*

1861, Dec. 10. Deane, William King. 25 *George-street.*

DATE OF ELECTION.

1879, Mar. 18. Dent, Hastings Charles. 20 *Thurloe-square, London, S.W.*

1883, Oct. 2. Faraday, Frederick James, F.L.S. *Brazennose Club.*

1840, Jan. 21. Gaskell, Rev. William, M.A. 46 *Plymouth Grove.*
1881, Nov. 1. Greg, Arthur. *Eagley, near Bolton.*
1874, Nov. 3. Grimshaw, Harry, F.C.S. *Thornton View, Clayton.*
1875, Feb. 9. Gwyther, R. F., M.A., Lecturer on Mathematics, Owens College. *Owens College.*

1878, Apr. 30. Harland, William Dugdale, F.C.S. 25 *Acomb-street, Greenheys.*
1862, Nov. 4. Hart, Peter. 192 *London-road.*
1839, Jan. 22. Hawkshaw, Sir John, F.R.S., F.G.S., Mem. Inst. C.E. 33 *Great George-street, Westminster, London, S.W.*
1873, Dec. 16. Heelis, James. 71 *Princess-street.*
1828, Oct. 31. Henry, William Charles, M.D., F.R.S. 11 *East-street, Lower Mosley-street.*
1833, Apr. 26. Heywood, James, F.R.S., F.G.S., F.S.A. 26 *Kensington-Palace-Gardens, London, W.*
1864, Mar. 22. Heywood, Oliver. *Bank, St. Ann's-street.*
1881, Nov. 1. Higgin, Alfred James. 22 *Little Peter-street, Gaythorn.*
1851, Apr. 29. Higgin, James. *Little Peter-street, Gaythorn.*
1884, Jan. 8. Hodgkinson, Alexander, M.B., D.Sc. *Claremont, Higher Broughton.*
1846, Jan. 27. Holden, James Platt. 3 *Temple Bank, Smedley Lane, Cheetham.*
1882, Oct. 17. Holt, Henry. *Fairlea, Didsbury.*
1884, Jan. 8. Hopkinson, Charles. 29 *Princess-street.*
1873, Dec. 2. Howorth, Henry H., F.S.A. *Derby House, Eccles.*
1884, Jan. 8. Hurst, Charles Herbert. *Owens College.*

1872, Feb. 6. Jewsbury, Sidney. 39 *Princess-street.*
1870, Nov. 1. Johnson, William H., B.Sc. 26 *Lever-street.*
1878, Nov. 26. Jones, Francis, F.R.S.E., F.C.S. *Grammar School.*
1848, Apr. 18. Joule, Benjamin St. John Baptist. 12 *Wardle-road, Sale.*
1842, Jan. 25. Joule, James Prescott, D.C.L., LL.D., F.R.S., F.C.S., Hon. Mem. C.P.S., and Inst. Eng. Scot., Corr. Mem. Inst. Fr. (Acad. Sc.) Paris, and Roy. Acad. Sc. Turin. 12 *Wardle-road, Sale.*

DATE OF ELECTION.

1852, Jan. 27. Kennedy, John Lawson. 47 *Mosley-street.*

1862, Apr. 29. Knowles, Andrew. *Swinton Old Hall, Swinton.*

1884, Jan. 8. Larmuth, Leopold. *Owens College.*

1863, Dec. 15. Leake, Robert, M.P. 75 *Princess-street.*

1850, Apr. 30. Leese, Joseph. *Fallowfield.*

1884, Jan. 22. London, Rev. Herbert, M.A. *The Rectory, High Leigh, Cheshire.*

1857, Jan. 27. Longridge, Robert Bentink. *Yew-Tree House, Tabley, Knutsford.*

1870, Apr. 19. Lowe, Charles. 43 *Piccadilly.*

1850, Apr. 30. Lund, Edward, F.R.C.S. Eng., L.S.A. 22 *St. John-street.*

1866, Nov. 13. McDougall, Arthur. *City Flour Mills, Poland-street.*

1859, Jan. 25. Maclure, John William, F.R.G.S. *Cross-street.*

1875, Jan. 26. Mann, John Dixon, M.D., M.R.C.P. Lond. 16 *St. John-street.*

1879, Dec. 2. Marshall, Alfred Milnes, M.A., D.Sc., Professor of Zoology, Owens College. *Owens College.*

1873, Nov. 4. Marshall, Rev. William, B.A. 81 *High-street, Chorlton-on-Medlock.*

1864, Nov. 1. Mather, William. *Iron Works, Deal-street, Brown-street, Salford.*

1873, Mar. 18. Melvill, James Cosmo, M.A., F.L.S. *Kersal Cottage, Prestwich.*

1879, Dec. 30. Millar, John Bell, B.E., Assistant Lecturer in Engineering, Owens College. *Owens College.*

1881, Oct. 18. Mond, Ludwig. *Winnington Hall, Northwich.*

1877, Nov. 27. Moore, Samuel. 25 *Dover-street, Chorlton-on-Medlock.*

1861, Oct. 29. Morgan, John Edward, M.B., M.A., M.R.C.P. Lond., F.R. Med. and Chir. S. 1 *St. Peter's-square.*

1850, Jan. 24. Newall, Henry. *Hare Hill, Littleborough.*

1873, Mar. 4. Nicholson, Francis, F.Z.S. 62 *Fountain-street.*

1862, Dec. 30. Ogden, Samuel. 10 *Mosley-street West.*

1861, Jan. 22. O'Neill, Charles, F.C.S., Corr. Mem. Ind. Soc. Mulhouse. 72 *Denmark-road.*

1844, Apr. 30. Ormerod, Henry Mere, F.G.S. 5 *Clarence-street.*

1861, Apr. 30. Parlane, James. *Rusholme.*

1876, Nov. 28. Parry, Thomas. *Grafton-place, Ashton-under-Lyne.*

1881, Nov. 29. Peacock, Richard. *Gorton Hall, Manchester.*

DATE OF ELECTION.

1874, Jan. 13. Pennington, Rooke, LL.B., F.G.S. 14 *Acresfield, Bolton.*

1854, Jan. 24. Pochin, Henry Davis. *Bodnant Hall, Conway.*

1861, Jan. 22. Radford, William. 1 *Princess-street.*

1854, Feb. 7. Ramsbottom, John. *Fern Hill, Alderley Edge.*

1859, Apr. 19. Ransome, Arthur, M.A., M.D. Cantab., M.R.C.S. 1 *St. Peter's-square.*

1869, Nov. 16. Reynolds, Osborne, M.A., F.R.S., Professor of Engineering, Owens College. *Ladybarn-road, Fallowfield.*

1883, Apr. 3. Rhodes, James, M.R.C.S. *Glossop.*

1880, Mar. 23. Roberts, Lloyd, M.D. *Kersal Towers, Higher Broughton.*

1860, Jan. 24. Roberts, William, M.D., B.A., F.R.S., M.R.C.P. Lond. 89 *Mosley-street.*

1864, Dec. 27. Robinson, John. *Atlas Works, Great Bridgewater-street.*

1822, Jan. 25. Robinson, Samuel. *Blackbrook Cottage, Wilmslow.*

1858, Jan. 26. Roscoe, Henry Enfield, B.A., LL.D., F.R.S., F.C.S., Professor of Chemistry, Owens College. *Owens College.*

1851, Apr. 29. Sandeman, Archibald, M.A. *Garry Cottage, near Perth.*

1870, Dec. 13. Schorlemmer, Carl, F.R.S., F.C.S. *Owens College.*

1842, Jan. 25. Schunck, Edward, Ph.D., F.R.S., F.C.S. *Oaklands, Kersal.*

1873, Nov. 18. Schuster, Arthur, Ph.D., F.R.S. *Owens College.*

1881, Nov. 29. Schwabe, Edmund Salis, B.A. 41 *George-street.*

1852, Apr. 20. Sidebotham, Joseph, F.R.A.S. *The Beeches, Bowdon.*

1876, Nov. 28. Smith, James. 35 *Cleveland-road, Crumpsall.*

1844, Apr. 29. Smith, Robert Angus, Ph.D., LL.D., F.R.S., F.C.S., Corr. Mem. Royal Bavarian Soc. 22 *Devonshire-street, All Saints.*

1884, Jan. 8. Smithells, Arthur, B.Sc. *Owens College.*

1859, Jan. 25. Sowler, Thomas. 24 *Cannon-street.*

1870, Nov. 1. Stewart, Balfour, LL.D., F.R.S., Professor of Natural Philosophy, Owens College. *Owens College.*

1863, Oct. 6. Stretton, Bartholomew. *Bridgewater-place, High-street.*

1884, Jan. 8. Swanwick, Frederick Tertius. *Owens College.*

1884, Mar. 18. Thomson, Alderman Joseph. *Wilmslow.*

1873, Apr. 15. Thomson, William, F.R.S.E., F.C.S. *Royal Institution.*

1860, Apr. 17. Trapp, Samuel Clement. 88 *Mosley-street.*

DATE OF ELECTION.

1882, Nov. 14. Ward, Harry Marshall, M.A., Berkeley Fellow. *Owens College.*

1879, Dec. 30. Ward, Thomas. *Brookfield House, Northwich.*

1873, Nov. 18. Waters, Arthur William, F.G.S. *Woodbrook, Alderley Edge.*

1874, Dec. 15. Watson, Morrison, M.D., Professor of Descriptive Anatomy, Owens College. *Linda Villa, Heald-road, Bowdon.*

1874, Jan. 27. Watts, John, Ph.D. 47 *Spring Gardens.*

1857, Jan. 27. Webb, Thomas George. *Glass Works, Kirby-street, Ancoats.*

1858, Jan. 26. Whitehead, James, M.D., M.R.C.P. Lond., F.R.C.S. Engl., L.S.A., M.R.I.A., Corr. Mem. Soc. Nat. Phil. Dresden, Med. Chir. Soc. Zurich, and Obst. Soc. Edin., Mem. Obst. Soc. Lond. *Royal Hotel, Manchester.*

1839, Jan. 22. Whitworth, Sir Joseph, Bart., F.R.S. *Chorlton-street, Portland-street.*

1859, Jan. 25. Wilde, Henry. *Rockside, Alderley Edge.*

1859, Apr. 19. Wilkinson, Thomas Read. *Manchester and Salford Bank, Mosley-street.*

1874, Nov. 3. Williams, William Carleton, F.C.S., Professor of Chemistry, Firth's College, Sheffield. *Firth's College, Sheffield.*

1851, Apr. 29. Williamson, William Crawford, F.R.S., Professor of Nat. Hist., Anat., and Physiol., Owens College, M.R.C.S. Engl., L.S.A. *Egerton-road, Fallowfield.*

1836, Jan. 22. Wood, William Rayner. *Singleton Lodge, near Manchester.*

1860, Apr. 17. Woolley, George Stephen. 69 *Market-street.*

1863, Nov. 17. Worthington, Samuel Barton, C.E. 12 *York-place, Oxford-street.*

1865, Feb. 21. Worthington, Thomas. 110 *King-street.*

1864, Nov. 1. Wright, William Cort, F.C.S. *Oakfield, Poynton. Cheshire.*

N.B.—Of the above list the following have compounded for their subscriptions, and are therefore Life Members:—

Brogden, Henry.
Johnson, William H., B.Sc.
Sandeman, Archibald, M.A.
Smith, Robert Angus, Ph.D., LL.D., F.R.S.

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